Recent advances
on prime graphs of integral group rings

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Joint work with V. Bovdi, E. Jespers, W. Kimmerle, S. Linton, S. Siciliano et al.

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Let $V = V(\mathbb{Z}G)$ be the normalised unit group of the integral group ring $\mathbb{Z}G$ of a finite group $G$.

The following conjectures about elements of $V$ state that:

- **(ZC-1):** they are rationally conjugate to elements of $G$ (Zassenhaus, 1974).
- **(IP-C):** they have same orders as elements of $G$.
- **(PQ):** if $G$ has elements of prime orders $p$ and $q$ but no elements of order $pq$, then $V$ has no elements of order $pq$ (Kimmerle, Oberwolfach Reports, 2007).
(ZC-1) holds for nilpotent groups (Weiss, 1991)

(PQ) holds for solvable groups (Kimmerle, 2006)

Clearly, (ZC-1) ⇒ (IP-C) ⇒ (PQ)

For a particular group $G$, all three conjectures involve looking at various possible orders of torsion units of $V(\mathbb{Z}G)$

Motivated by this, jointly with V. Bovdi we started a project of determination of properties of torsion units of $V(\mathbb{Z}G)$ for sporadic simple groups
Partial augmentations
sums of coefficients of elements of $\mathbb{Z}G$ over conjugacy classes

Criterion for ZC-1
formulated in terms of partial augmentations

HeLP (Hertweck-Luthar-Passi) method
uses character tables to produce constraints on partial augmentations
Reducing to a finite problem

Finitely many possible orders of torsion units

The order of $u \in V(\mathbb{Z}G)$ divides $\exp(G)$
(Cohn–Livingstone, 1965)

Finitely many possible partial augmentations

$$\nu_i(u)^2 \leq |C_i| \quad \text{and} \quad \sum_{i=1}^{n} \frac{\nu_i(u)^2}{|C_i|} \leq 1$$
(Hales–Luthar–Passi, 1990)

- $|\nu_{5a}| \leq 39$ for $M_{11}$
- $|\nu_{30a}| \leq 58023609591071951707573011$ for $M$
Reducing the number of search variables

- 10 conjugacy classes of elements in $M_{11}$
- 194 in $M$

Berman–Higman Theorem (1955)

$$\text{tr}(u) = \nu_1 = 0$$


$$\nu_g(u) \neq 0 \Rightarrow o(g) \text{ divides } o(u)$$

- Now only 2 search variables for order 10 in $M_{11}$
- ... but 28 search variables for order 30 in the Monster
Main source of constraints

Theorem (Luthar–Passi, 1989; modular case - Hertweck, 2005)

for all $l$ the number

$$\mu_l(u, \chi, p) = \frac{1}{k} \sum_{d \mid k} \frac{\text{Tr}_{Q(z^d)/Q}(\chi(u^d)z^{-dl})}{Q(\chi(u^d)z^{-dl})}$$

is a non-negative integer which is not greater than $\deg(\chi)$, where:

- $p$ is either 0 or a prime divisor of $|G|$
- $u \in V(\mathbb{Z}G)$ is a normalized torsion unit of order $k$
- if $p \neq 0$, then $k$ and $p$ must be coprime
- $z$ is a complex primitive $k$-th root of unity
- $\chi$ is a classical character or a $p$-Brauer character of $G$
Example: order 77 for $Co_1$

$$\mu_{11}(u, \chi_2, 0) = \frac{1}{77} (-100\nu_7 a - 30\nu_7 b - 10\nu_{11} a + 10\nu_{11} (7) - 10\nu_{7} (11) - 3\nu_{7} (11) + 276) \geq 0;$$

$$\mu_0(u, \chi_3, 0) = \frac{1}{77} (300\nu_7 a + 300\nu_7 b + 120\nu_{11} a + 20\nu_{11} (7) + 30\nu_{7} (11) + 30\nu_{7} (11) + 299) \geq 0;$$

$$\mu_0(u, \chi_4, 0) = \frac{1}{77} (840\nu_7 a + 84\nu_{7} (11) + 1771) \geq 0;$$

$$\mu_1(u, \chi_4, 0) = \frac{1}{77} (14\nu_7 a - 14\nu_{7} (11) + 1771) \geq 0;$$

$$\mu_7(u, \chi_4, 0) = \frac{1}{77} (-84\nu_7 a + 84\nu_{7} (11) + 1771) \geq 0;$$

$$\mu_{11}(u, \chi_4, 0) = \frac{1}{77} (-140\nu_7 a - 14\nu_{7} (11) + 1771) \geq 0;$$

$$\mu_0(u, \chi_7, 0) = \frac{1}{77} (840\nu_7 a - 120\nu_{11} a - 20\nu_{11} (7) + 84\nu_{7} (11) + 27300) \geq 0;$$

$$\mu_0(u, \chi_{15}, 13) = \frac{1}{77} (-420\nu_7 a + 60\nu_{11} a + 10\nu_{11} (7) - 42\nu_{7} (11) + 474145) \geq 0;$$

$$\nu_7 a + \nu_7 b + \nu_{11} a = 1; \quad \nu_{7} (11) + \nu_{7} (11) = 1 \quad \nu_{11} (7) = 1;$$
Results

(PQ) holds for 13 sporadic simple groups:

- $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}$
- $J_1, J_2, J_3$
- $HS, McL, He, Ru, Suz$

Furthermore:

- For $G = ON$, the prime graph of $V(\mathbb{Z}G)$ is not connected
- For $G = Co_3, Co_2$ and $Co_1$, prime graphs of $G$ and $V(\mathbb{Z}G)$ have the same number of components

Recent overview:

Hot off the press

- (PQ) may be reduced to the examination of nonabelian composition factors and their automorphism groups.
- Theorem (W. Kimmerle, AK, 2012): (PQ) holds for a group of order divisible by three 3 primes, except possibly the case when $M_{10}$ or $PGL(2, 9)$ are involved in $G$.
- A. Bächle and L. Margolis (2013) completed the remaining two cases of $M_{10}$ and $PGL(2, 9)$.
- Theorem (W. Kimmerle, AK, 2013): If $G$ is one of $M_{12}$, $M_{22}$, $J_2$, $J_3$, $HS$, $McL$, $He$, $Suz$, then (PQ) holds for $Aut(G)$
- Theorem (W. Kimmerle, AK, 2013): (PQ) holds for a finite group provided that each its composition factor $S$ is isomorphic to one of the 13 sporadic simple groups for which (PQ) holds, or $|S|$ divisible by at most three primes.
Some challenges

for Co3, this will give a positive answer to (PQ): 

$$|u| = 35 \Rightarrow (\nu_5^a, \nu_5^b, \nu_7^a) \not\in \{(3, 12, -14), (4, 11, -14)\}$$

for $M_{11}$, this will solve (IP-C):

$$|u| = 12 \Rightarrow (\nu_2^a, \nu_4^a, \nu_6^a) \not\in \{(-1, 1, 1), (1, 1, -1)\}$$

Complete all Brauer character tables for Co1
and hope to solve (PQ) by eliminating orders 55 and 65

Complete all Brauer character tables for $J_4$
and cut about $2^{54}$ admissible triples $(\nu_{31}^a, \nu_{31}^b, \nu_{31}^c)$

Find a counterexample to the 1st Zassenhaus conjecture
constructing a unit with prescribed partial augmentations