On the orders of elements in almost simple groups with excceptional socle

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Groups St Andrews in Birmingham

Definitions

$\omega(G)$ — the set of orders of the elements of G, or its spectrum.

Groups are isospectral if their spectra coincide.

$h(G)$ — the number of pairwise non-isomorphic groups isospectral to G.

G is recognizable by its spectrum if $h(G) = 1$, i.e. for any group H

$$\omega(H) = \omega(G) \Rightarrow G \simeq H.$$

Recognition by spectrum problem is solved for a group G if we know $h(G)$ (and if $h(G)$ is finite then the groups isospectral to G are determined).

Recognition problem for simple groups

Main goal

To solve recognition problem for all nonabelian finite simple groups.

Theorem (2015)

Let S be one of the following nonabelian simple groups:

- Sporadic groups other than J_2
- Alternating groups other than A_6 and A_{10}
- Exceptional groups of Lie type other than ${}^3D_4(2)$
- $PSL_n(q)$ and $PSU_n(q)$ with $n \geqslant 45$
- $PSp_{2n}(q)$, $\Omega_{2n+1}(q)$ and $P\Omega^{\pm}_{2n}(q)$ with $n \geqslant 31$.

If G is a finite group having the same set of the orders of elements as S then $S \leqslant G \leqslant \operatorname{Aut} S$.

Almost simple groups isospectral to their socles

Problem

For all nonabelian finite simple groups S, determine all groups G such that $\omega(G) = \omega(S)$ and $S \leqslant G \leqslant \operatorname{Aut} S$.

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Theorem (Zavarnitsine, 2004)

Let $S = PSL_3(q)$, where $q = p^m > 3$, p is an odd prime. Then finite groups isospectral to S are precisely:

- S if $q \equiv 3, 11 \pmod{12}$;
- S and $S \rtimes \langle\gamma\rangle$ if $q \equiv 5, 9 \pmod{12}$;
- G with $S \leqslant G \leqslant S \rtimes \langle\varphi\rangle$, where φ is a field automorphism of S of order $(m)_3$ (the highest power of 3 dividing m) if $q \equiv 1 \pmod 6$.

Exceptional groups

- ${}^2B_2(q), {}^2G_2(q), {}^2F_4(q)$ — recognizable (Brandl, Shi, Deng, 1992–1999)
- $G_2(3^m)$ — recognizable (Vasil'ev, 2002)
- $F_4(2^m)$ — recognizable (Vasil'ev, Mazurov, Shi, ..., 2005)
- $E_8(q)$ — recognizable (Kondrat'ev, 2010)
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Does there exist a finite group G isospectral to a finite simple exceptional group S of Lie type, but G is not isomorphic to S?

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If S is one of the remaining groups and $\omega(G) = \omega(S)$, then $S \leqslant G \leqslant \operatorname{Aut} S$. In particular, $h(S)$ is finite.

Main results

- ${}^3D_4(q)$, $F_4(q)$ (Grechkoseeva, Z., 2015)
- $E_6^{\varepsilon}(q)$, $E_7(q)$ (Z., 2016)

Theorem 1

Let $S = F_4(q)$, $q = p^m$, and $S < G \leqslant \mathrm{Aut}(S)$. Then $\omega(G) = \omega(S)$ iff G/S is a 2-group, and $p \notin \{2, 3, 7, 11\}$.

Theorem 2

Let $S = {}^3D_4(q)$, $q = p^m$, and $S < G \leqslant \mathrm{Aut}(S)$. Then $\omega(G) = \omega(S)$ iff G/S is a 2-group, and $p \geqslant 7$.

Theorem 3

Let $S = E_7(q)$, $q = p^m$, and $S < G \leqslant \mathrm{Aut}\, S$. Then $\omega(G) = \omega(S)$ iff G is an extension of S by a field automorphism, G/S is a 2-group и $p \notin \{2, 13, 17\}$.

Main results

Notation: $\varepsilon \in \{+, -\}$, $E_6^+(q) = E_6(q)$, $E_6^-(q) = {}^2E_6(q)$.

Theorem 4

Let $S = E_6^\varepsilon(q)$, where q is a power of a prime p, and $S < G \leqslant \operatorname{Aut} S$. Then $\omega(G) = \omega(S)$ if and only if G is an extension of S by a field automorphism, G/S is a 3-group, 3 divides $q - \varepsilon 1$, and $p \notin \{2, 11\}$.

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Theorem 4

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Example

If $S = E_6(5^6)$, $S < G \leqslant \operatorname{Aut} S$ and $\omega(G) = \omega(S)$, then $G \simeq S \rtimes \langle \varphi \rangle$, where φ is a field automorphism of S of order 3. In particular, $h(S) = 2$.

Let S be a simple exceptional group of Lie type ${}^dX_n(q)$, where $q = p^m$, p is a prime. Then $h(S)$ is as indicated in Table 1. If $1 < h(S) < \infty$, then a finite group is isospectral to S if and only if it is isomorphic to a group G such that $S \leqslant G \leqslant S \rtimes \langle\varphi\rangle$, where φ is a field automorphism of a group S of the order given in the table.

S	Conditions	$\lvert\varphi\rvert$	$h(S)$
${}^2B_2(q)$		$-$	1
${}^2G_2(q)$		$-$	1
${}^2F_4(q)$		$-$	1
$G_2(q)$		$-$	1
$E_8(q)$		$-$	1
${}^3D_4(q)$	$p \notin \{2, 3, 7, 11\}$, $(m)_2 = 2^s \geqslant 2$	2^s	$s+1$
	($p \in \{2, 3, 7, 11\}$ or m is odd) and $q \neq 2$	$-$	1
	$q = 2$	$-$	∞
$F_4(q)$	$p \notin \{2, 3, 7, 11\}$, $(m)_2 = 2^s \geqslant 2$	2^s	$s+1$
	otherwise	$-$	1
$E_6^{\varepsilon}(q)$	$p \notin \{2, 11\}$, $3 \mid q - \varepsilon 1$, $(m)_3 = 3^s \geqslant 3$	3^s	$s+1$
	otherwise	$-$	1
$E_7(q)$	$p \notin \{2, 13, 17\}$, $(m)_2 = 2^s \geqslant 2$	2^s	$s+1$
	otherwise	$-$	1