From minimal non-abelian subgroups to finite non-abelian $p$-groups

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Minimal non-abelian $p$-groups

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<tr>
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The structure of subgroups of $A_t$-groups

$G$ is an $A_t$-group

$A_0, A_1, A_2, \cdots, A_{t-2}, A_{t-1}$

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$\cdots$ \hspace{2cm} \cdots

$A_0, A_1, A_2$

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order

$p^n$

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All possible types of $A_t$-subgroups of order $p^{n-j}$ are $A_0, A_1, A_2, \cdots, A_{t-2}, A_{t-j}$ and $G$ has at least one $A_{t-j}$-subgroup for $j = 1, 2, \cdots, t, t \leq n - 2$. 
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- An $A_1$-group is exactly a minimal non-abelian $p$-group.
- Every finite $p$-group must be an $A_t$-group for some $t$. Hence the study of finite $p$-groups is equivalent to that of $A_t$-groups. In particular, if a finite $p$-group is of order $p^n$, then $t \leq n - 2$. 
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Qu et al. classified finite $p$-groups which are a center extension of a cyclic $p$-group, and elementary abelian $p$-groups by a minimal non-abelian $p$-group, respectively. Their results were contained in the following four papers.
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Although we use the results of classification mentioned above, the classification of $A_3$-groups is still an enormous work. The classification provide many useful information to the study of $p$-groups. Some new results are discovered and proved, and some new problems are proposed.
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The sketch of the classification of $A_3$-groups are showed as follows.
How is $A_3$-groups classified?

The sketch of the classification of $A_3$-groups

$G$ is an $A_3$-groups having an $A_1$-subgroup of index $p$

$G$ has an abelian subgroup of index $p$  $G$ has no abelian subgroup of index $p$

$d(G) = 2$  $d(G) = 3$

6 types ([7])  20 types ([5])

$G$ has at least two $A_1$-subgroups of index $p$

$G$ has a unique $A_1$-subgroup of index $p$

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17 types ([7])  19 types ([6])

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Facts and Problems

We observed that

- $A_2$-groups are the $p$-groups all of whose $A_1$-subgroups are of index $p$. 

Moreover, Berkovich and Janko in their book “Groups of Prime Power Order Vol. 2” proposed the following Problem [Problem 920]. Classify the $p$-groups all of whose $A_1$-subgroups are of order $p^3$. 

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For convenience, we use $M_p(2, 1)$ to denote the metacyclic $p$-group of order $p^3$, and $M_p(1, 1, 1)$ the non-metacyclic $p$-group of order $p^3$, respectively.
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The results of classification

**Theorem (Q.H. Zhang).** Assume $G$ is a finite nonabelian $p$-group with $d(G) = n$, $p$ an odd prime. Then all $A_1$-subgroups of $G$ are isomorphic to $M_p(1, 1, 1)$ if and only if $G$ is one of the following groups:

1. nonabelian groups with $\exp(G) = p$;
2. $G = H_p \rtimes \langle a \rangle$, a semidirect product of $H_p$ and $\langle a \rangle$, where $H_p = B_1 \times B_2 \times \cdots \times B_{n-1}$ is an abelian Hughes subgroup of index $p$, $a^p = 1$. Moreover, $\langle B_i, a \rangle$ is a groups of maximal class with an abelian subgroup of index $p$ and whose union elements are of order $p$, or an elementary abelian group of order $p^2$, where $i = 1, 2, \ldots, n - 1$. 
The structure of subgroups of $p$-groups we classified

$G$ is an $A_t$-group

\[ A_{t-1}, \ldots, A_{t-1}, A_0, \ldots, A_0 \]

\[ A_{t-2}, \ldots, A_{t-2}, A_0, \ldots, A_0 \]

\[ \ldots \]

\[ A_1, \ldots, A_1, A_0, \ldots, A_0 \]

\[ A_2, \ldots, A_2, A_0, \ldots, A_0 \]

\[ A_{t-2}, \ldots, A_{t-2}, A_0, \ldots, A_0 \]

\[ A_{t-1}, \ldots, A_{t-1}, A_0, \ldots, A_0 \]

\[ A_t, \ldots, A_t, A_0, \ldots, A_0 \]

order

\[ p^n \]

\[ p^{n-1} \]

\[ p^{n-2} \]

\[ \ldots \]

\[ p^4 \]

\[ p^3 \]
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<table>
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<td>$A_{t-2}, \cdots, A_{t-2}, A_0, \cdots, A_0$</td>
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</tr>
<tr>
<td>\ldots \ldots \ldots</td>
<td>\ldots</td>
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<td>$A_2, \cdots, A_2, A_0, \cdots, A_0$</td>
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All possible types of $A_i$-subgroups of order $p^{n-j}$ are $A_0$ and $A_{t-j}$ and $G$ has at least one $A_{t-j}$-subgroup for $j = 1, 2, \cdots, t-1$, $t \leq n - 2$. 
The structure of subgroups of $A_t$-groups and more

In addition, my colleagues have also classified finite $p$-groups with the structure of subgroups showed as follows.

$G$ is an $A_t$-group

$A_{t-1}, \cdots, A_{t-1}, A_0(\leq p)$

$A_{t-2}, \cdots, A_{t-2}, A_0(\leq p)$

$\cdots \cdots \cdots \cdots$

$A_2, \cdots, A_2, A_0(\leq p)$

$A_1, \cdots, A_1, A_0(\leq p)$

$A_0, \cdots, A_0, A_0$

\begin{align*}
\text{order} & \quad \text{order} \\
p^n & \quad p^{n-1} \\
p^{n-1} & \quad p^{n-2} \\
\cdots & \quad \cdots \\
p^{n-(t-2)} & \quad p^{n-(t-1)} \\
p^{n-t} &
\end{align*}
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- $A_{t-2}, \cdots, A_{t-2}, A_0(\leq p)$
- $\cdots$
- $A_{2}, \cdots, A_{2}, A_0(\leq p)$
- $A_{1}, \cdots, A_{1}, A_0(\leq p)$
- $A_0, \cdots, A_0, A_0$
- $\text{order}$
  - $p^n$
  - $p^{n-1}$
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The structure of subgroups of ordinary metacyclic $p$-groups

Qu et al. in [J. Algebra Appl. 13:4(2014)] classified finite $p$-groups with the structure of subgroups showed as follows.

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- $A_{t-1}$
- $A_{t-2}$
- ..............
- $A_2$
- $A_1$
- $A_0$

order

- $p^n$
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order

\[
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A_{t-1} & \quad p^n \\
A_{t-2} & \quad p^{n-1} \\
\cdots & \quad \cdots \\
A_2 & \quad p^{n-(t-2)} \\
A_1 & \quad p^{n-(t-1)} \\
A_0 & \quad p^{n-t}
\end{align*}
\]

It turns out that such $p$-groups are exactly ordinary metacyclic $p$-groups.
Qu et al. in [J. Algebra Appl. 13:4(2014)] classified finite $p$-groups with the structure of subgroups showed as follows.

<table>
<thead>
<tr>
<th>Order</th>
<th>$p^n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_t$-group</td>
<td>$p^n$</td>
</tr>
<tr>
<td>$A_{t-1}$</td>
<td>$p^{n-1}$</td>
</tr>
<tr>
<td>$A_{t-2}$</td>
<td>$p^{n-2}$</td>
</tr>
<tr>
<td>.............</td>
<td>...</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$p^{n-(t-2)}$</td>
</tr>
<tr>
<td>$A_1$</td>
<td>$p^{n-(t-1)}$</td>
</tr>
<tr>
<td>$A_0$</td>
<td>$p^{n-t}$</td>
</tr>
</tbody>
</table>

It turns out that such $p$-groups are exactly ordinary metacyclic $p$-groups.

Such $p$-groups can be regarded as the $p$-groups “with least possible types of $A_i$-subgroups”.
The structure of subgroups of ordinary metacyclic $p$-groups

Qu et al. in [J. Algebra Appl. 13:4 (2014)] classified finite $p$-groups with the structure of subgroups showed as follows.

<table>
<thead>
<tr>
<th>$G$ is an $A_t$-group</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{t-1}$</td>
<td>$p^n$</td>
</tr>
<tr>
<td>$A_{t-2}$</td>
<td>$p^{n-1}$</td>
</tr>
<tr>
<td>$A_{t-3}$</td>
<td>$p^{n-2}$</td>
</tr>
<tr>
<td>$\ldots$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$p^{n-(t-2)}$</td>
</tr>
<tr>
<td>$A_1$</td>
<td>$p^{n-(t-1)}$</td>
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The structure of subgroups of $A_t$-groups and more

My colleagues Zhang et al. have classified finite $p$-groups with the structure of subgroups showed as follows.

\begin{align*}
G \text{ is an } A_t\text{-group} \\
A_0, A_1, A_2, \cdots, A_{t-2}, A_{t-1} \\
A_0, A_1, A_2, \cdots, A_{t-2} \\
\ldots \\
A_0, A_1, A_2 \\
A_0, A_1 \\
A_0 \\
\end{align*}

| Order | $p^n$ | $p^{n-1}$ | $p^{n-2}$ | $p^{n-(t-2)}$ | $p^{n-(t-1)}$ | $p^{n-t}$ |
The structure of subgroups of $A_t$-groups and more

My colleagues Zhang et al. have classified finite $p$-groups with the structure of subgroups showed as follows.

$G$ is an $A_t$-group

<table>
<thead>
<tr>
<th>Order</th>
<th>$A_0, A_1, A_2, \cdots, A_{t-2}, A_{t-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$A_0, A_1, A_2, \cdots, A_{t-2}$</td>
</tr>
<tr>
<td></td>
<td>\ldots</td>
</tr>
<tr>
<td></td>
<td>$A_0, A_1$</td>
</tr>
<tr>
<td></td>
<td>$A_0$</td>
</tr>
</tbody>
</table>

Such $p$-groups can be regarded as the $p$-groups "with most possible types of $A_t$-subgroups".
From minimal non-abelian subgroups to finite non-abeian $p$-groups

Qinhai Zhang

Members of $p$-group team of Shanxi Normal University
Thank you!