Factorisations of almost simple groups

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joint work with Cai Heng Li

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Let $G$ be a group and $H, K$ be subgroups of $G$. If $G = HK$, then we call the expression $G = HK$ a factorisation of $G$ and $H, K$ the factors of this factorisation. If both factors are maximal subgroups, then the factorisation is said to be maximal.

A group $G$ is said to be almost simple if $L \leq G \leq \text{Aut}(L)$ for some finite nonabelian simple group $L$. In this case, $L$ is the unique minimal normal subgroup of $G$, which is thus the socle of $G$.

Group factorisations arise in many contexts besides group theory: symmetry of Cayley graphs, graph embedding on surfaces, 2-arc-transitive digraphs... Factorisations of almost simple groups is at the heart of study of group factorisations.
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Let $G$ be an almost simple group with $L \leq G \leq \text{Aut}(L)$, where $L$ is a nonabelian simple group.

Hering, Liebeck and Saxl (1987) classified factorisations of $G$ if $L$ is exceptional Lie type.

Liebeck, Praeger and Saxl (1990) classified all factorisations of $G$ if $L$ is alternating.


For the case where $L$ is classical, the maximal factorisations of $G$ was classified by Liebeck, Praeger and Saxl (1990), but a classification of all the factorisations is still largely open.
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Our work

We classify factorisations of almost simple groups for the following two cases:

1. At least one factor is soluble.
2. At least one factor has at least two insoluble composition factors.

Some special cases of (1) have been classified in the literature. This motivated our work in case (i).

Combination of (1) and (2) reduces the problem of factorisations of almost simple groups to the case where both factors have a unique insoluble composition factor.
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Background/Motivation
Influence of factors on the structure of factorized group

Suppose $G = HK$ is a factorisation.

If $H$, $K$ are both abelian, then $G$ is meta-abelian. (Ito 1955)

If $H$, $K$ are both nilpotent, then $G$ is soluble. (Wielandt 1958 and Kegel 1961)

If $H$, $K$ are both soluble, then the possible insoluble composition factors of $G$ are:

- $\text{PSL}_2(q)$,
- $\text{PSL}_3(q)$ with $q \in \{3, 4, 5, 7, 8\}$,
- $\text{PSL}_4(2)$,
- $\text{PSU}_3(8)$,
- $\text{PSp}_4(3)$, and
- $M_{11}$.

(Kazarin 1986)

The crucial step in the proof of Kazarin's result was to determine which almost simple groups have a factorisation with both factors soluble.
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The crucial step in the proof of Kazarin’s result was to determine which almost simple groups have a factorisation with both factors soluble.
Carlitz’s conjecture

Let $q$ be a power of a prime $p$, and $f(x) \in \mathbb{F}_q[x]$. We say $f(x)$ is an exceptional polynomial if there are infinitely many finite extensions $K$ of $\mathbb{F}_q$ such that $t \mapsto f(t)$ is a permutation of $K$.

Carlitz in 1966 conjectured that if $p$ is odd, then every exceptional polynomial must have odd degree.

In the proof of Carlitz’s conjecture (Fried, Guralnick and Saxl 1993), it is shown that any counterexample with minimal degree $n$ would give a factorisation $G = HK$ of an almost simple group $G$ such that $H$ is maximal of index $n$ in $G$ and $K/O_p(K)$ is cyclic with some additional condition.

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Given a group $G$ and a subset $S$ of $G$ such that $1/\mathbb{Z} \in S$ and $S - 1 = S$, the graph $(V, E)$ defined by $V = G$ and $E = \{\{g, sg\} | g \in G, s \in S\}$ is called the Cayley graph of $G$ with respect to $S$.

Let $\Gamma = (V, E)$ be a regular graph. For a positive integer $s$, an $s$-arc of $\Gamma$ is a sequence of vertices $(v_0, v_1, \ldots, v_s)$ such that $\{v_i - 1, v_i\} \in E$ for $1 \leq i \leq s$ and $v_i - 1 \neq v_{i+1}$ for $1 \leq i \leq s - 1$.

$\Gamma$ is said to be $s$-arc-transitive if $\text{Aut}(\Gamma)$ acts transitively on the set of $s$-arcs of $\Gamma$.

There are classification results on 2-arc-transitive Cayley graphs of dihedral groups (Du, Malniˇc, Marušiˇc) and abelian groups (Ivanov, Praeger; Li, Pan).

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Main result
Factorisations of almost simple groups with a soluble factor

Let $G$ be an almost simple group with socle $L$, and $G = HK$ be a factorisation with none of $H$, $K$ contains $L$.

Theorem (C. H. Li and B. X. 2015+)
If $L$ is classical Lie type with $H$ soluble and $K$ insoluble, then with finitely many exceptions which are explicitly known, the triple $(L, H \cap L, K \cap L)$ lies in the table next slide.

The case where $L$ is not classical can be read off from the classifications of all the factorisations. The case where both $H$ and $K$ are soluble is easy to work out based on Kazarin's result.
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<thead>
<tr>
<th>$L$</th>
<th>$H \cap L \leq$</th>
<th>$K \cap L \geq$</th>
<th>remark</th>
</tr>
</thead>
<tbody>
<tr>
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<td>$q^{m^2} : \frac{q^{2m-1}}{(q+1)d} \cdot m &lt; P_m$</td>
<td>$\text{SU}_{2m-1}(q)$</td>
<td>$m \geq 2$; $d = (2m, q + 1)$</td>
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Theorem (C. H. Li and B. X. 2017+)

If $H$ has at least two unsolvable composition factors, then either (a) or (b) below holds.

(a) $A_n \leq H \leq S_n$ with $n \geq 10$, and one of the following holds:
   (a.1) $H$ is a transitive permutation group of degree $n$, and $A_n - 1 \leq K \leq S_n - 1$;
   (a.2) $n = 10$, $H$ is a transitive permutation group of degree $10$ such that $(A_5 \times A_5) 2 \leq H \leq S_5 \wr S_2$, and $K = \text{SL}_2(8)$ or $\text{SL}_2(8)$ 3;
   (a.3) $n = 12$, $A_7 \times A_5 \leq H \leq S_7 \times S_5$, and $K = M_{12}$;
   (a.4) $n = 24$, $A_{19} \times A_5 \leq H \leq S_{19} \times S_5$, and $K = M_{24}$.

(b) $(L, H \cap L, K \cap L)$ lies in the table next slide, where $Q \leq 2$.
Factorisations of almost simple groups with a “very insoluble” factor

Let $G$ be an almost simple group with socle $L$, and $G = HK$ be a factorisation with none of $H, K$ contains $L$.
Factorisations of almost simple groups with a "very insoluble" factor

Let $G$ be an almost simple group with socle $L$, and $G = HK$ be a factorisation with none of $H, K$ contains $L$.

**Theorem (C. H. Li and B. X. 2017+)**

*If $H$ has at least two unsolvable composition factors, then either (a) or (b) below holds.*

(a) $A_n \leq G \leq S_n$ with $n \geq 10$, and one of the following holds:

(a.1) $H$ is a transitive permutation group of degree $n$, and $A_{n-1} \leq K \leq S_{n-1}$;

(a.2) $n = 10$, $H$ is a transitive permutation group of degree 10 such that $(A_5 \times A_5).2 \leq H \leq S_5 \wr S_2$, and $K = SL_2(8)$ or $SL_2(8).3$;

(a.3) $n = 12$, $A_7 \times A_5 \leq H \leq S_7 \times S_5$, and $K = M_{12}$;

(a.4) $n = 24$, $A_{19} \times A_5 \leq H \leq S_{19} \times S_5$, and $K = M_{24}$.

(b) $(L, H \cap L, K \cap L)$ lies in the table next slide, where $Q \leq 2$. 
<table>
<thead>
<tr>
<th>row</th>
<th>$L$</th>
<th>$H \cap L$</th>
<th>$K \cap L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\text{Sp}_{4\ell}(2^f), f\ell \geq 2$</td>
<td>$(\text{Sp}<em>{2a}(2^{fb}) \times \text{Sp}</em>{2a}(2^{fb})).R.2$, $ab = \ell$ and $R \leq b \times b$</td>
<td>$\Omega_{4\ell}^-(2^f).Q$</td>
</tr>
<tr>
<td>2</td>
<td>$\text{Sp}_{12\ell}(2^f)$</td>
<td>$(G_2(2^{f\ell}) \times G_2(2^{f\ell})).R.2$, $R \leq \ell \times \ell$</td>
<td>$\Omega_{12\ell}^-(2^f).Q$</td>
</tr>
<tr>
<td>3</td>
<td>$\text{Sp}_{4\ell}(4), \ell \geq 2$</td>
<td>$\text{Sp}<em>2(4) \times \text{Sp}</em>{4\ell-2}(4)$</td>
<td>$\text{Sp}_{2\ell}(16).Q$</td>
</tr>
<tr>
<td>4</td>
<td>$\text{Sp}_{4\ell}(4), \ell \geq 2$</td>
<td>$(\text{Sp}<em>2(4) \times \text{Sp}</em>{2\ell}(4)).P$, $P \leq 2$</td>
<td>$\Omega_{4\ell}^-(4).Q$</td>
</tr>
<tr>
<td>5</td>
<td>$\text{Sp}_{8\ell}(2)$</td>
<td>$(\text{Sp}<em>2(4) \times \text{Sp}</em>{2\ell}(4)).P$, $P \leq [8]$</td>
<td>$\Omega_{8\ell}^-(2).Q$</td>
</tr>
<tr>
<td>6</td>
<td>$\text{Sp}_4(2^f), f \geq 3$ odd</td>
<td>$(\text{Sp}_2(2^f) \times \text{Sp}_2(2^f)).P$, $P \leq 2$</td>
<td>$\text{Sz}(2^f)$</td>
</tr>
<tr>
<td>7</td>
<td>$\text{Sp}_6(2^f), f \geq 2$</td>
<td>$U \times \text{Sp}_4(2^f)$, $U$ unsolvable and $U \leq \text{Sp}_2(2^f)$</td>
<td>$G_2(2^f)$</td>
</tr>
<tr>
<td>8</td>
<td>$\text{Sp}_{12}(4)$</td>
<td>$\text{Sp}<em>2(4) \times \text{Sp}</em>{10}(4)$</td>
<td>$G_2(16).Q$</td>
</tr>
<tr>
<td>9</td>
<td>$\text{Sp}_{12}(4)$</td>
<td>$(\text{Sp}_2(4) \times G_2(4)).P$, $P \leq 2$</td>
<td>$\Omega_{12}^-(4).Q$</td>
</tr>
<tr>
<td>10</td>
<td>$\text{Sp}_{24}(2)$</td>
<td>$(\text{Sp}_2(4) \times G_2(4)).P$, $P \leq [8]$</td>
<td>$\Omega_{24}^-(2).Q$</td>
</tr>
<tr>
<td>11</td>
<td>$\text{P}\Omega_{4\ell}^+(q), \ell \geq 2$, $q \geq 4$</td>
<td>$(U \times \text{PSp}<em>{2\ell}(q)).P$, $U$ unsolvable, $U \leq \text{PSp}</em>{2}(q)$ and $P \leq \gcd(2, \ell, q - 1)$</td>
<td>$\Omega_{4\ell-1}(q)$</td>
</tr>
<tr>
<td>12</td>
<td>$\Omega_{8\ell}^+(2)$</td>
<td>$(\text{Sp}<em>2(4) \times \text{Sp}</em>{2\ell}(4)).P$, $P \leq 2^2$</td>
<td>$\text{Sp}_{8\ell-2}(2)$</td>
</tr>
<tr>
<td>13</td>
<td>$\Omega_{8\ell}^+(4)$</td>
<td>$(\text{Sp}<em>2(4^c) \times \text{Sp}</em>{2\ell}(16)).P$, $c \leq 2$ and $P \leq 2^2$</td>
<td>$\text{Sp}_{8\ell-2}(4)$</td>
</tr>
<tr>
<td>14</td>
<td>$\Omega_{12}^+(2^f), f \geq 2$</td>
<td>$U \times G_2(2^f)$, $U$ unsolvable and $U \leq \text{Sp}_2(2^f)$</td>
<td>$\text{Sp}_{10}(2^f)$</td>
</tr>
<tr>
<td>15</td>
<td>$\Omega_{24}^+(2)$</td>
<td>$(\text{Sp}_2(4) \times G_2(4)).P$, $P \leq 2^2$</td>
<td>$\text{Sp}_{22}(4)$</td>
</tr>
<tr>
<td>16</td>
<td>$\Omega_{24}^+(4)$</td>
<td>$(\text{Sp}_2(4^c) \times G_2(16)).P$, $c \leq 2$ and $P \leq 2^2$</td>
<td>$\text{Sp}_{22}(4)$</td>
</tr>
</tbody>
</table>
Thank you for listening!