Factorisations of almost simple groups

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joint work with Cai Heng Li

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- Factorisations of almost simple groups is at the heart of study of group factorisations.

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- Giudici (2006) classified factorisations of G if L is sporadic simple.
- For the case where L is classical, the maximal factorisations of G was classified by Liebeck, Praeger and Saxl (1990), but a classification of all the factorisations is still largely open.

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(2) At least one factor has at least two insoluble composition factors.

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- Combination of (1) and (2) reduces the problem of factorisations of almost simple groups to the case where both factors have a unique insoluble composition factor.

Background/Movitation

Influence of factors on the structure of factorized group

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- If H, K are both abelian, then G is meta-abelian. (Ito 1955)
- If H, K are both nilpotent, then G is soluble. (Wielandt 1958 and Kegel 1961)
- If H, K are both soluble, then the possible insoluble composition factors of G are: $\mathrm{PSL}_2(q)$, $\mathrm{PSL}_3(q)$ with $q \in \{3, 4, 5, 7, 8\}$, $\mathrm{PSL}_4(2)$, $\mathrm{PSU}_3(8)$, $\mathrm{PSp}_4(3)$ and M_{11}. (Kazarin 1986)

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The crucial step in the proof of Kazarin's result was to determine which almost simple groups have a factorisation with both factors soluble.

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In the proof of Carlitz's conjecture (Fried, Guralnick and Saxl 1993), it is shown that any counterexample with minimal degree n would give a factorisation $G = HK$ of an almost simple group G such that H is maximal of index n in G and $K/\mathbf{O}_p(K)$ is cyclic with some additional condition.

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Given a group G and a subset S of G such that $1 \notin S$ and $S^{-1} = S$, the graph (V, E) defined by $V = G$ and $E = \{\{g, sg\} \mid g \in G,\ s \in S\}$ is called the Cayley graph of G with respect to S.

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Let $\Gamma = (V, E)$ be a regular graph. For a positive integer s, an s-arc of Γ is a sequence of vertices $(v_0, v_1, \ldots, v_s)$ such that $\{v_{i-1}, v_i\} \in E$ for $1 \leqslant i \leqslant s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leqslant i \leqslant s - 1$.

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There are classification results on 2-arc-transitive Cayley graphs of dihedral groups (Du, Malnič, Marušič) and abelian groups (Ivanov, Praeger; Li, Pan).

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To classify 2-arc-transitive Cayley graphs of soluble groups needs to know factorisations of almost simple groups with a soluble factor.

Main result

Factorisations of almost simple groups with a soluble factor

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Let G be an almost simple group with socle L, and $G = HK$ be a factorisation with none of H, K contains L.

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Theorem (C. H. Li and B. X. 2015+)

If L is classical Lie type with H soluble and K insoluble, then with finitely many exceptions which are explicitly known, the triple $(L, H \cap L, K \cap L)$ lies in the table next slide.

- The case where L is not classical can be read off from the classifications of all the factorisations.

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- The case where L is not classical can be read off from the classifications of all the factorisations.
- The case where both H and K are soluble is easy to work out based on Kazarin's result.

Table 1

L	$H \cap L \leqslant$	$K \cap L \trianglerighteq$	remark
$\mathrm{PSL}_n(q)$	$\hat{}\mathrm{GL}_1(q^n){:}n = \frac{q^n-1}{(q-1)d}{:}n$	$q^{n-1}{:}\mathrm{SL}_{n-1}(q)$	$d = (n, q-1)$
$\mathrm{PSL}_4(q)$	$q^3{:}\frac{q^3-1}{d}.3 < \mathrm{P}_1$ or P_3	$\mathrm{PSp}_4(q)$	$d = (4, q-1)$
$\mathrm{PSU}_{2m}(q)$	$q^{m^2}{:}\frac{q^{2m}-1}{(q+1)d}.m < \mathrm{P}_m$	$\mathrm{SU}_{2m-1}(q)$	$m \geqslant 2$ $d = (2m, q+1)$
$\mathrm{PSp}_{2m}(q)$	$q^{m(m+1)/2}{:}(q^m-1).m < \mathrm{P}_m$	$\Omega^-_{2m}(q)$	$m \geqslant 2$, q even
$\mathrm{PSp}_4(q)$	$q^3{:}(q^2-1).2 < \mathrm{P}_1$	$\mathrm{Sp}_2(q^2)$	q even
$\mathrm{PSp}_4(q)$	$q^{1+2}{:}\frac{q^2-1}{2}.2 < \mathrm{P}_1$	$\mathrm{PSp}_2(q^2)$	q odd
$\Omega_{2m+1}(q)$	$(q^{m(m-1)/2}.q^m){:}\frac{q^m-1}{2}.m < \mathrm{P}_m$	$\Omega^-_{2m}(q)$	$m \geqslant 3$, q odd
$\mathrm{P}\Omega^+_{2m}(q)$	$q^{m(m-1)/2}{:}\frac{q^m-1}{d}.m < \mathrm{P}_{m-1}$ or P_m	$\Omega_{2m-1}(q)$	$m \geqslant 5$ $d = (4, q^m-1)$
$\mathrm{P}\Omega^+_8(q)$	$q^6{:}\frac{q^4-1}{d}.4 < \mathrm{P}_1$, P_3 or P_4	$\Omega_7(q)$	$d = (4, q^4-1)$

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Theorem (C. H. Li and B. X. 2017+)

If H has at least two unsolvable composition factors, then either (a) or (b) below holds.

(a) $\mathrm{A}_n \leqslant G \leqslant \mathrm{S}_n$ *with* $n \geqslant 10$, *and one of the following holds:*
- (a.1) *H is a transitive permutation group of degree n, and* $\mathrm{A}_{n-1} \leqslant K \leqslant \mathrm{S}_{n-1}$;
- (a.2) $n = 10$, *H is a transitive permutation group of degree* 10 *such that* $(\mathrm{A}_5 \times \mathrm{A}_5).2 \leqslant H \leqslant \mathrm{S}_5 \wr \mathrm{S}_2$, *and* $K = \mathrm{SL}_2(8)$ *or* $\mathrm{SL}_2(8).3$;
- (a.3) $n = 12$, $\mathrm{A}_7 \times \mathrm{A}_5 \leqslant H \leqslant \mathrm{S}_7 \times \mathrm{S}_5$, *and* $K = \mathrm{M}_{12}$;
- (a.4) $n = 24$, $\mathrm{A}_{19} \times \mathrm{A}_5 \leqslant H \leqslant \mathrm{S}_{19} \times \mathrm{S}_5$, *and* $K = \mathrm{M}_{24}$.

(b) $(L, H \cap L, K \cap L)$ *lies in the table next slide, where* $Q \leqslant 2$.

Table 2

row	L	$H \cap L$	$K \cap L$
1	$\mathrm{Sp}_{4\ell}(2^f)$, $f\ell \geqslant 2$	$(\mathrm{Sp}_{2a}(2^{fb}) \times \mathrm{Sp}_{2a}(2^{fb})).R.2$, $ab = \ell$ and $R \leqslant b \times b$	$\Omega^-_{4\ell}(2^f).Q$
2	$\mathrm{Sp}_{12\ell}(2^f)$	$(\mathrm{G}_2(2^{f\ell}) \times \mathrm{G}_2(2^{f\ell})).R.2$, $R \leqslant \ell \times \ell$	$\Omega^-_{12\ell}(2^f).Q$
3	$\mathrm{Sp}_{4\ell}(4)$, $\ell \geqslant 2$	$\mathrm{Sp}_2(4) \times \mathrm{Sp}_{4\ell-2}(4)$	$\mathrm{Sp}_{2\ell}(16).Q$
4	$\mathrm{Sp}_{4\ell}(4)$, $\ell \geqslant 2$	$(\mathrm{Sp}_2(4) \times \mathrm{Sp}_{2\ell}(4)).P$, $P \leqslant 2$	$\Omega^-_{4\ell}(4).Q$
5	$\mathrm{Sp}_{8\ell}(2)$	$(\mathrm{Sp}_2(4) \times \mathrm{Sp}_{2\ell}(4)).P$, $P \leqslant [8]$	$\Omega^-_{8\ell}(2).Q$
6	$\mathrm{Sp}_4(2^f)$, $f \geqslant 3$ odd	$(\mathrm{Sp}_2(2^f) \times \mathrm{Sp}_2(2^f)).P$, $P \leqslant 2$	$\mathrm{Sz}(2^f)$
7	$\mathrm{Sp}_6(2^f)$, $f \geqslant 2$	$U \times \mathrm{Sp}_4(2^f)$, U unsolvable and $U \leqslant \mathrm{Sp}_2(2^f)$	$\mathrm{G}_2(2^f)$
8	$\mathrm{Sp}_{12}(4)$	$\mathrm{Sp}_2(4) \times \mathrm{Sp}_{10}(4)$	$\mathrm{G}_2(16).Q$
9	$\mathrm{Sp}_{12}(4)$	$(\mathrm{Sp}_2(4) \times \mathrm{G}_2(4)).P$, $P \leqslant 2$	$\Omega^-_{12}(4).Q$
10	$\mathrm{Sp}_{24}(2)$	$(\mathrm{Sp}_2(4) \times \mathrm{G}_2(4)).P$, $P \leqslant [8]$	$\Omega^-_{24}(2).Q$
11	$\mathrm{P}\Omega^+_{4\ell}(q)$, $\ell \geqslant 2$, $q \geqslant 4$	$(U \times \mathrm{PSp}_{2\ell}(q)).P$, U unsolvable, $U \leqslant \mathrm{PSp}_2(q)$ and $P \leqslant \gcd(2, \ell, q-1)$	$\Omega_{4\ell-1}(q)$
12	$\Omega^+_{8\ell}(2)$	$(\mathrm{Sp}_2(4) \times \mathrm{Sp}_{2\ell}(4)).P$, $P \leqslant 2^2$	$\mathrm{Sp}_{8\ell-2}(2)$
13	$\Omega^+_{8\ell}(4)$	$(\mathrm{Sp}_2(4^c) \times \mathrm{Sp}_{2\ell}(16)).P$, $c \leqslant 2$ and $P \leqslant 2^2$	$\mathrm{Sp}_{8\ell-2}(4)$
14	$\Omega^+_{12}(2^f)$, $f \geqslant 2$	$U \times \mathrm{G}_2(2^f)$, U unsolvable and $U \leqslant \mathrm{Sp}_2(2^f)$	$\mathrm{Sp}_{10}(2^f)$
15	$\Omega^+_{24}(2)$	$(\mathrm{Sp}_2(4) \times \mathrm{G}_2(4)).P$, $P \leqslant 2^2$	$\mathrm{Sp}_{22}(4)$
16	$\Omega^+_{24}(4)$	$(\mathrm{Sp}_2(4^c) \times \mathrm{G}_2(16)).P$, $c \leqslant 2$ and $P \leqslant 2^2$	$\mathrm{Sp}_{22}(4)$

Thank you for listening!