Realizing Saturated Fusion Systems

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August 12, 2017

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Infinite families of exotic fusion systems

Example (Infinite families of exotic fusion systems)

Let $r = 2k + 1 \geq 5$ be odd. Let B be a rank two 3-group of order 3^r with the presentation

$$B = \langle s, s_1, .., s_{r-1} \mid s_i = [s_{i-1}, s], [s_i, s_1] = s_j^3 s_{j+1}^3 s_{j+2} = s^3 = 1 \rangle$$

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- $\omega : B \to B : s \mapsto s^{-1}, s_1 \mapsto s_1^2 s_2 \quad \eta : B \to: s \mapsto s, s_1 \mapsto s_1^{-1}$.

Theorem (Alperin)

Let $\mathcal{F}$ be a saturated fusion system over a p-group T. Then $\mathcal{F} = \langle Aut_{\mathcal{F}}(T), Aut_{\mathcal{F}}(P) \mid P \text{ is } \mathcal{F}\text{-essential in } T \rangle$

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Theorem (Park, '10)

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Question: What is this G, and how do we construct it?

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Lemma

Let ϕ, ψ be two maps inside T. Then

$$|\mathcal{O}_\phi^\psi| = \frac{|N_{\psi,\phi}||C_T(I_\psi)|}{|D_\phi|} \leq \frac{|N_T(D_\psi, D_\phi)||C_T(I_\psi)|}{|D_\phi|}$$

where $N_{\psi,\phi} = \{x \in T \mid \exists y \in T \text{ with } (D_\psi)^x \leq D_\phi, \text{ and } c_x|_{D_\psi} \circ \phi \circ c_y = \psi\}$.

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Question: How do we construct Ω?

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$$\Omega = \bigsqcup_{\phi \in \Gamma} C_1(\phi) \cdot \mathcal{O}_\phi,$$

where $C_1(\phi) = \sum_{\psi \in \Gamma_\phi} C(\psi) \geq 0$.

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Theorem (Park, '10)

Let $\mathcal{F}$ be a fusion system [saturated fusion system] over a finite p-group T. Let Ω be a right semicharacteristic set [characteristic set] corresponding to $\mathcal{F}$. Define G to be a group of permutations of Ω that preserve the action on the right in the following way:

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- $e(\mathcal{F}) \neq 0 \Leftrightarrow \mathcal{F}$ is exotic.

Theorem (Diaz, Ruiz, Viruel)

Let $\mathcal{F}$ be a saturated fusion system over B with at least one proper $\mathcal{F}$-essential subgroup. Then the outer automorphism group of the $\mathcal{F}$-essential subgroups are as follows:

T	V_0	E_0	E_1	E_{-1}	A
$\langle \omega \rangle$		$SL_2(3)$			
$\langle \omega \rangle$			$SL_2(3)$	$SL_2(3)$	
$\langle \omega \rangle$		$SL_2(3)$	$SL_2(3)$	$SL_2(3)$	
$\langle \eta \rangle$					$SL_2(3)$
$\langle \omega\eta \rangle$	$SL_2(3)$				
$\langle \eta, \omega \rangle$					$GL_2(3)$
$\langle \eta, \omega \rangle$			$SL_2(3)$		
$\langle \eta, \omega \rangle$			$SL_2(3)$		$GL_2(3)$
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$\langle \eta, \omega \rangle$		$GL_2(3)$	$SL_2(3)$		
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$\langle \eta, \omega \rangle$	$GL_2(3)$				$GL_2(3)$
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Let $\mathcal{F} = \langle \mathrm{Aut}_{\mathcal{F}}(T), \mathrm{Aut}_{\mathcal{F}}(E_0) \rangle$ with $\mathrm{Out}_{\mathcal{F}}(T) \cong \langle \omega \rangle$ and $\mathrm{Out}_{\mathcal{F}}(V_0) \cong \mathrm{SL}_2(3)$ be a saturated fusion system over B. Then the **minimal** characteristic set is given by

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where $n_k = 3^{2k-3} - 1$ and the maps

$$\theta_0 : E_0 \to E_0 : s \mapsto s_1^{3^{k-1}}; s_1^{3^{k-1}} \mapsto s^{-1}$$

Infinite families of exotic fusion systems

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- $k = 4 \implies e(\mathcal{F}) \lessapprox 1.0 \times 10^7$

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- The exoticity index satisfies $e(\mathcal{F}) \leq 30k + 21$.
- $k = 2 \implies e(\mathcal{F}) \leq 81$.

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- The exoticity index satisfies $e(\mathcal{F}) \leq 30k + 21$.
- $k = 2 \implies e(\mathcal{F}) \leq 81$.
- $k = 3 \implies e(\mathcal{F}) \leq 111$.

Example (2)

Let $\mathcal{F} = \langle \mathrm{Aut}_{\mathcal{F}}(T), \mathrm{Aut}_{\mathcal{F}}(A) \rangle$ with $\mathrm{Out}_{\mathcal{F}}(T) \cong \langle \omega, \eta \rangle$ and $\mathrm{Out}_{\mathcal{F}}(A) \cong \mathrm{GL}_2(3)$ be a saturated fusion system over B. Then the **minimal** characteristic set is given by

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- $\mathcal{F}$ is realizable, via $G \cong A \rtimes \mathrm{GL}_2(3) \implies e(\mathcal{F}) = 0$.

Example (3)

Let $\mathcal{F} = \langle \mathrm{Aut}_{\mathcal{F}}(T), \mathrm{Aut}_{\mathcal{F}}(V_0) \rangle$, with $\mathrm{Out}_{\mathcal{F}}(T) \cong \langle \omega, \eta \rangle$ and $\mathrm{Out}_{\mathcal{F}}(V_0) \cong \mathrm{GL}_2(3)$ be a saturated fusion system over B. Then the **minimal** characteristic set is given by

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where $m_k = 3^{2k-2} - 1$ and the maps:

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- $\mathcal{F}$ is realizable, $\implies e(\mathcal{F}) = 0$.

Theorem (Diaz, Ruiz, Viruel)

Let $\mathcal{F}$ be a saturated fusion system over B with at least one proper $\mathcal{F}$-essential subgroup. Then the outer automorphism group of the $\mathcal{F}$-essential subgroups are as follows:

	T	V_0	E_0	E_1	E_{-1}	A	$\vert\Omega\vert/\vert T\vert$
✓	$\langle\omega\rangle$		$SL_2(3)$				$2(3^{2k-2}-1)^2$
✓	$\langle\omega\rangle$			$SL_2(3)$	$SL_2(3)$		$2[2\cdot 3^{2k-2}(3^{2k-2}-2)+1]$
✓	$\langle\omega\rangle$		$SL_2(3)$	$SL_2(3)$	$SL_2(3)$		$2[3^{2k-1}(3^{2k-2}-2)+1]$
	$\langle\eta\rangle$					$SL_2(3)$	
	$\langle\omega\eta\rangle$	$SL_2(3)$					
✓	$\langle\eta,\omega\rangle$					$GL_2(3)$	2^4
✓	$\langle\eta,\omega\rangle$			$SL_2(3)$			$2^2[2^3(3^{2k-2})^2-2^2\cdot 3^{2k-2}+1]$
	$\langle\eta,\omega\rangle$			$SL_2(3)$		$GL_2(3)$	
✓	$\langle\eta,\omega\rangle$		$GL_2(3)$				$2^2(3^{2k-2}-1)^2$
	$\langle\eta,\omega\rangle$		$GL_2(3)$			$GL_2(3)$	
	$\langle\eta,\omega\rangle$		$GL_2(3)$	$SL_2(3)$			
	$\langle\eta,\omega\rangle$		$GL_2(3)$	$SL_2(3)$		$GL_2(3)$	
✓	$\langle\eta,\omega\rangle$	$GL_2(3)$					$4(3^{2k-1}-1)^2$
	$\langle\eta,\omega\rangle$	$GL_2(3)$				$GL_2(3)$	
✓	$\langle\eta,\omega\rangle$	$GL_2(3)$		$SL_2(3)$			$2^2[2^3\cdot 3^{8k-4}-2^4\cdot 3^{6k-2}+17\cdot 3^{4k-4}-2^3\cdot 3^{2k-2}+1]$
	$\langle\eta,\omega\rangle$	$GL_2(3)$		$SL_2(3)$		$GL_2(3)$	

Thank You For Listening!