Degree of commutativity of infinite groups

... or how I learnt about rational growth and ends of groups

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The concept of degree of commutativity was first introduced by Erdős and Turán (1968) and Gustafson (1973) for finite groups:

Definition 1.1

Let F be a finite group. The *degree of commutativity* of F is

$$\mathrm{dc}(F) := \frac{|\{(x,y)\in F^2 \mid xy=yx\}|}{|F|^2} = \frac{\sum_{x\in F} |C_F(x)|}{|F|^2}, \qquad (1)$$

where $C_F(x)$ is the centraliser of x in F.

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Examples

- F is abelian if and only if $\mathrm{dc}(F) = 1$.
- In fact, F is abelian whenever $\mathrm{dc}(F) > \frac{5}{8}$. Indeed, $\mathrm{dc}(F) = k/|F|$, where k is the number of conjugacy classes in F, and the center of a group cannot have index 2 or 3.
- This bound is sharp: for $F = D_8$ (dihedral group of order 8), $\mathrm{dc}(F) = \frac{5}{8}$.

This concept has recently been generalised to all finitely generated groups (Antolín, Martino, Ventura, 2015):

Definition 1.2

Let G be a finitely generated group and X a finite generating set. The *degree of commutativity* of G with respect to X is

$$\mathrm{dc}_X(G) := \limsup_{n\to\infty} \frac{|\{(x,y)\in B_X(n)^2 \mid xy=yx\}|}{|B_X(n)|^2} \qquad (2)$$

where $B_X(n)$ is the ball of radius n in the Cayley graph $\mathrm{Cay}(G,X)$.

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Conjecture 1.3 (Antolín, Martino, Ventura, 2015)

- $\mathrm{dc}_X(G) = 0$ *whenever* G *is not virtually abelian.*
- $\mathrm{dc}_X(G) \le \frac{5}{8}$ *whenever* G *is not abelian.*

In particular, (conjecturally) $\mathrm{dc}_X(G) = 0$ whenever G has exponential growth.

Consider intermediate cases between free and free abelian groups:

Definition 2.1

Let Δ be a finite simple graph. One can define a group G_Δ, called the *right-angled Artin group* associated with Δ, as a group given by the presentation

$$G_\Delta := \langle V(\Delta) \mid xy = yx \text{ for all } xy \in E(\Delta) \rangle. \quad (3)$$

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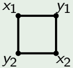
Remark

The same is true for exponentially growing groups with some torsion – i.e. if relations $x^{m(x)} = 1$ for $x \in V(\Delta)$ are added to the presentation.

Example

If $\Delta =$ 

, then $G = G_\Delta \cong F_2(X) \times F_2(Y)$ where $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$. Any element in $F_2(X)$ commutes with any element in $F_2(Y)$, and

$$|B_{X\cup Y}(n)| \sim 8n3^{n-1}, \quad \text{and} \tag{4}$$

$$|F_2(X) \cap B_{X\cup Y}(n)| = |F_2(Y) \cap B_{X\cup Y}(n)| \sim 4 \times 3^{n-1}. \tag{5}$$

It follows that

$$\frac{|\{(x,y) \in B_{X\cup Y}(n)^2 | xy = yx\}|}{|B_{X\cup Y}(n)|^2} \geq \frac{|F_2(X \text{ or } Y) \cap B_{X\cup Y}(n)|^2}{|B_{X\cup Y}(n)|^2} \sim \frac{1}{4n^2}. \tag{6}$$

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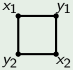
If $\Delta =$ 

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Thus arguments comparing the exponential growth rates are not enough... We need some sort of "fine counting" of elements in balls.

Definition 2.3

Let G be a group with a finite generating set X. The *growth series* of G with respect to X is

$$s_{G,X}(t) := \sum_{g \in G} t^{|g|_X} = \sum_{n=0}^{\infty} |S_X(n)| t^n \in \mathbb{Z}[[t]]. \tag{7}$$

G is said to be of *rational growth* with respect to X if $s_{G,X}(t)$ is a rational function of t, i.e. $s_{G,X}(t) = \frac{p(t)}{q(t)}$ for some polynomials p, q.

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This is relevant because:

Theorem 2.4 (Chiswell, 1994)

Let Δ be a finite simple graph. Then $s_{G_\Delta, V(\Delta)}(t)$ is rational.

Theorem 2.5 (Valiunas, 2016)

Let G be an infinite group with a finite generating set X, and suppose $s_{G,X}(t)$ is a rational function. Then there exist constants $\alpha \in \mathbb{Z}_{\geq 1}$, $\lambda \in [1, \infty)$ and $D > C > 0$ such that

$$Cn^{\alpha-1}\lambda^n \leq |S_X(n)| \leq Dn^{\alpha-1}\lambda^n \tag{8}$$

for all $n \geq 1$.

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The equality $\mathrm{dc}_{V(\Delta)}(G_\Delta) = 0$ then can be derived from the fact that otherwise we can find two disjoint subsets of $V(\Delta)$ generating subgroups "comparable in size" to G. This follows from:

Theorem 2.6 (Servatius, 1989)

Let $g \in G_\Delta$ be an element such that $|g|_{V(\Delta)} \leq |p^{-1}gp|_{V(\Delta)}$ for any $p \in G_\Delta$. Then $C_G(g) \cong \mathbb{Z}^\ell \times \langle W \rangle$ where $W \subseteq V(\Delta)$ and g can be written using only letters of $V(\Delta) \setminus W$.

Another generalisation of free groups comes from considering groups with "sufficiently tree-like" Cayley graphs.

Definition 3.1

- For a locally compact graph Γ, define the *number of ends* $e(\Gamma)$ of Γ to be the supremum of the number of unbounded connected components of $\Gamma \setminus K$, where K ranges over all compact subsets of Γ.
- If G is a group with a finite generating set X, the *number of ends* of G with respect to X is defined to be

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Examples

- If G is finite, then $\mathrm{Cay}(G, X)$ is bounded, so $e_X(G) = 0$.
- If G is virtually $\mathbb{Z}$, then $\mathrm{Cay}(G, X)$ is quasi-isometric to $\mathbb{R}$, so $e_X(G) = 2$.

Examples (continued)

- If $G = \mathbb{Z}^m$ for $m \geq 2$ and X are the standard generators, then $\mathrm{Cay}(G, X)$ is an m-dimensional "grid", and we can see that $e_X(G) = 1$.
- If $G = F_m$ for $m \geq 2$ and X is a free basis, then $\mathrm{Cay}(G, X)$ is a $2m$-regular tree, so $e_X(G) = \infty$.

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The following associates $e_X(G)$ with algebraic structure of G:

Theorem 3.2 (Stallings, 1971)

Let G be a group with a finite generating set X. Then $e_X(G) > 1$ if and only if G admits an edge-transitive action on a tree T with finite edge stabilisers and without globally fixed points. Moreover, $e_X(G) = 2$ if T is a line, and $e_X(G) = \infty$ otherwise.

In particular, $e(G) = e_X(G)$ is independent of the set X.

The action of G on T can be used to show:

Theorem 3.3 (Valiunas, 2016)

Let G be a finitely generated group with infinitely many ends, and let X be any finite generating set. Then $\mathrm{dc}_X(G) = 0$.

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Let $e \in E(T)$ be an edge and let $H_1, H_2 \leq G$ be the stabilisers of its endpoints. Let $\mathcal{E} := \bigcup_{g \in G} H_1^g \cup \bigcup_{g \in G} H_2^g \subseteq G$ be the set of *elliptic* elements of G, i.e. elements that fix some vertex in T. The proof of the Theorem relies on the following:

Lemma 3.4 (Valiunas, 2016; Yang, 2017)

$\mathcal{E}$ is negligible in G, i.e. $\frac{|\mathcal{E} \cap B_X(n)|}{|B_X(n)|} \to 0$ as $n \to \infty$.

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Remark

Similar argument works more generally – for non-elementary relatively hyperbolic groups.

Thank you!