Unipotent automorphisms of solvable groups

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Remarks. (1) Let $H = G \rtimes \mathrm{Aut}(G)$. The element $a \in \mathrm{Aut}(G)$ is unipotent (n-unipotent) if and only if a is a left Engel (n-Engel) element in $G\langle a \rangle$.

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(2) G is an Engel (n-Engel) group if and only if all the elements in $\mathrm{Inn}\,(G)$ are unipotent (n-unipotent).

Definition. Let G be group with a finite series of subgroups

$$G = G_0 \geq G_1 \geq \cdots \geq G_m = \{1\}.$$

The stability group of the series is the subgroup S of Aut (G) consisting of the automorphisms a where for each $i = 1, \ldots, m$ and $g \in G_{i-1}$ we have $[g, a] \in G_i$.

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Theorem (W. Burnside) Let V be a a finite dimensional vector space and $H \leq \mathrm{GL}(V)$ where H consists of unipotent automorphisms. Then H stabilises a finite series of subspaces
$V = V_0 \geq V_1 \geq \cdots \geq V_m = \{0\}$.

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Theorem(Frati 2014). Let G be a solvable group and H a finitely generated nilpotent subgroup of Aut (G) consisting of n-unipotent automorphisms. Then H stabilizes a finite series in G. Moreover, the nilpotency class of H is (n, r)-bounded if H is generated by r elements.

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Theorem(Puglisi,& T 2017). Let G be a solvable group and H a solvable subgroup of Aut (G) whose elements are n-unipotent.

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Remark. In fact one can deduce the following stronger variant of the one given in last remark. If $\mathcal{P}_n$ is the set of prime divisors of $e(n)$ in the following result then it suffices that G is $\mathcal{P}_n$-torsion free.

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Theorem. (Crosby & T 2010). Let H be a normal right n-Engel subgroup of a group G that belongs to some term of the upper central series. Then there exist positive integers $c(n), e(n)$ such that

$$[H^{e(n)},_{c(n)} G] = [H,_{c(n)} G]^{e(n)}.$$

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Example 1. Let $G = A \operatorname{wr} H$ where A is a cyclic group of order 2 and H is a countable elementary abelian 2-group. The group $\operatorname{Inn}(G)$ then acts G as a group of 3-unipotent automorphisms but it still doesn't stabilize any finite series.

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Example 1. Let $G = A \operatorname{wr} H$ where A is a cyclic group of order 2 and H is a countable elementary abelian 2-group. The group $\operatorname{Inn}(G)$ then acts G as a group of 3-unipotent automorphisms but it still doesn't stabilize any finite series.

Example 2. Let m be the smallest positive integer such that the Burnside variety $\mathcal{B}(2^m)$ is not locally finite. Choose any finitely generated infinite group G in $\mathcal{B}(2^m)$. Every involution $a \in G$ induces an $(m+1)$-unipotent automorphism in $\operatorname{Inn}(G)$ and there must exist such an involution that does not stabilize a finite series in G. Otherwise we get the contradiction that G is finite.