Unipotent automorphisms of solvable groups

Gunnar Traustason

Department of Mathematical Sciences
University of Bath

Groups St Andrews 2017 in Birmingham
Unipotent automorphisms of solvable groups

1. Introduction.
2. Solvable groups acting $n$-unipotently on solvable groups.
3. Examples.
1. Introduction

Definition. Let $G$ be a group and $a$ an automorphism of $G$, we say that $a$ is a unipotent automorphism if for every $g \in G$ there exists a non-negative integer $n = n(g)$ such that $[g, n a], \ldots, [g, a] = 1$.

If $n(g) = n$ can be chosen independently of $g$, then we say that $a$ is an $n$-unipotent automorphism.

Remarks. (1) Let $H = G \rtimes \text{Aut}(G)$. The element $a \in \text{Aut}(G)$ is unipotent ($n$-unipotent) if and only if $a$ is a left Engel ($n$-Engel) element in $G \langle a \rangle$.

(2) $G$ is an Engel ($n$-Engel) group if and only if all the elements in $\text{Inn}(G)$ are unipotent ($n$-unipotent).

Gunnar Traustason

Unipotent automorphisms of solvable groups
1. Introduction

Definition. Let $G$ be a group and $a$ an automorphism of $G$, we say that $a$ is a unipotent automorphism if for every $g \in G$ there exists a non-negative integer $n = n(g)$ such that

$$[g, na] = [[g, a], \ldots, a] = 1.$$
1. Introduction

Definition. Let $G$ be a group and $a$ an automorphism of $G$, we say that $a$ is a unipotent automorphism if for every $g \in G$ there exists a non-negative integer $n = n(g)$ such that

$$[g, n\ a] = [[[g, a], \ldots, a]]_{n} = 1.$$ 

If $n(g) = n$ can be chosen independently of $g$, then we say that $a$ is an $n$-unipotent automorphism.
1. Introduction

Definition. Let $G$ be a group and $a$ an automorphism of $G$, we say that $a$ is a unipotent automorphism if for every $g \in G$ there exists a non-negative integer $n = n(g)$ such that

$$[g, n\ a] = [[[g, a], \ldots, a] = 1.$$

If $n(g) = n$ can be chosen independently of $g$, then we say that $a$ is an $n$-unipotent automorphism.

Remarks. (1) Let $H = G \rtimes \text{Aut}(G)$. The element $a \in \text{Aut}(G)$ is unipotent ($n$-unipotent) if and only if $a$ is a left Engel ($n$-Engel) element in $G\langle a \rangle$. 
1. Introduction

Definition. Let $G$ be a group and $a$ an automorphism of $G$, we say that $a$ is a unipotent automorphism if for every $g \in G$ there exists a non-negative integer $n = n(g)$ such that

$$[g, n a] = [[g, a], \ldots, a] = 1.$$ 

If $n(g) = n$ can be chosen independently of $g$, then we say that $a$ is an $n$-unipotent automorphism.

Remarks. (1) Let $H = G \rtimes \text{Aut}(G)$. The element $a \in \text{Aut}(G)$ is unipotent ($n$-unipotent) if and only if $a$ is a left Engel ($n$-Engel) element in $G \langle a \rangle$.

(2) $G$ is an Engel ($n$-Engel) group if and only if all the elements in $\text{Inn}(G)$ are unipotent ($n$-unipotent).
Definition. Let $G$ be group with a finite series of subgroups

$$G = G_0 \geq G_1 \geq \cdots \geq G_m = \{1\}.$$  

The stability group of the series is the subgroup $S$ of $\text{Aut}(G)$ consisting of the automorphisms $a$ where for each $i = 1, \ldots, m$ and $g \in G_{i-1}$ we have $[g, a] \in G_i$. 

Theorem (P. Hall, 1958). The stability group $S$ of a subgroup series of length $m$ is nilpotent of class at most $\frac{m(m-1)}{2}$.

Theorem (W. Burnside) Let $V$ be a finite dimensional vector space and $H \leq \text{GL}(V)$ where $H$ consists of unipotent automorphisms. Then $H$ stabilises a finite series of subspaces $V = V_0 \geq V_1 \geq \cdots \geq V_m = \{0\}$. 

Gunnar Traustason  
Unipotent automorphisms of solvable groups
**Definition.** Let $G$ be a group with a finite series of subgroups

$$G = G_0 \geq G_1 \geq \cdots \geq G_m = \{1\}.$$  

The **stability group** of the series is the subgroup $S$ of $\text{Aut}(G)$ consisting of the automorphisms $a$ where for each $i = 1, \ldots, m$ and $g \in G_{i-1}$ we have $[g, a] \in G_i$.

**Theorem** (P. Hall, 1958). The stability group $S$ of a subgroup series of length $m$ is nilpotent of class at most $m(m - 1)/2$. 

Theorem (W. Burnside). Let $V$ be a finite dimensional vector space and $H \leq \text{GL}(V)$ where $H$ consists of unipotent automorphisms. Then $H$ stabilises a finite series of subspaces $V = V_0 \geq V_1 \geq \cdots \geq V_m = \{0\}$. 

Unipotent automorphisms of solvable groups
**Definition.** Let $G$ be a group with a finite series of subgroups

$$G = G_0 \geq G_1 \geq \cdots \geq G_m = \{1\}.$$ 

The **stability group** of the series is the subgroup $S$ of $\text{Aut}(G)$ consisting of the automorphisms $a$ where for each $i = 1, \ldots, m$ and $g \in G_{i-1}$ we have $[g, a] \in G_i$.

**Theorem** (P. Hall, 1958). The stability group $S$ of a subgroup series of length $m$ is nilpotent of class at most $m(m - 1)/2$.

**Theorem** (W. Burnside) Let $V$ be a finite dimensional vector space and $H \leq \text{GL}(V)$ where $H$ consists of unipotent automorphisms. Then $H$ stabilises a finite series of subspaces $V = V_0 \geq V_1 \geq \cdots \geq V_m = \{0\}$. 

Gunnar Traustason

Unipotent automorphisms of solvable groups
**Question**: Let $G$ be a group and $H$ a subgroup of $\text{Aut}(G)$ consisting of unipotent automorphisms. Is $H$ nilpotent? Moreover does $H$ stabilize a finite series for $G$?

**Remark.** If $H$ stabilizes a finite series in $G$, then $H$ is nilpotent. The converse is not true in general for a $n$-unipotent $H \leq \text{Aut}(G)$.

**Theorem** (Frati 2014). Let $G$ be a solvable group and $H$ a finitely generated nilpotent subgroup of $\text{Aut}(G)$ consisting of $n$-unipotent automorphisms. Then $H$ stabilizes a finite series in $G$. Moreover, the nilpotency class of $H$ is $(n, r)$-bounded if $H$ is generated by $r$ elements.
Question: Let $G$ be a group and $H$ a subgroup of $\text{Aut}(G)$ consisting of unipotent automorphisms. Is $H$ nilpotent? Moreover does $H$ stabilize a finite series for $G$?

Remark. If $H$ stabilizes a finite series in $G$, then $H$ is nilpotent. The converse is not true in general for a $n$-unipotent $H \leq \text{Aut}(G)$. 
Question: Let $G$ be a group and $H$ a subgroup of $\text{Aut}(G)$ consisting of unipotent automorphisms. Is $H$ nilpotent? Moreover does $H$ stabilize a finite series for $G$?

Remark. If $H$ stabilizes a finite series in $G$, then $H$ is nilpotent. The converse is not true in general for a $n$-unipotent $H \leq \text{Aut}(G)$.

Theorem (Frati 2014). Let $G$ be a solvable group and $H$ a finitely generated nilpotent subgroup of $\text{Aut}(G)$ consisting of $n$-unipotent automorphisms. Then $H$ stabilizes a finite series in $G$. Moreover, the nilpotency class of $H$ is $(n, r)$-bounded if $H$ is generated by $r$ elements.
**Question:** Let $G$ be a group and $H$ a subgroup of $\text{Aut}(G)$ consisting of unipotent automorphisms. Is $H$ nilpotent? Moreover does $H$ stabilize a finite series for $G$?
Question: Let $G$ be a group and $H$ a subgroup of $\text{Aut}(G)$ consisting of unipotent automorphisms. Is $H$ nilpotent? Moreover does $H$ stabilize a finite series for $G$?

Remark The question whether $H$ is nilpotent is open when $H$ is finite. (Casolo and Puglisi (2013) have though shown that $H$ is nilpotent whenever $H$ is finite and $G$ is locally graded.
Question: Let $G$ be a group and $H$ a subgroup of $\text{Aut}(G)$ consisting of unipotent automorphisms. Is $H$ nilpotent? Moreover does $H$ stabilize a finite series for $G$?

Remark The question whether $H$ is nilpotent is open when $H$ is finite. (Casolo and Puglisi (2013) have though shown that $H$ is nilpotent whenever $H$ is finite and $G$ is locally graded.

Theorem (Casolo & Puglisi 2013). If $G$ satisfies max and $H$ is a subgroup consisting of unipotent automorphisms, then $H$ stabilizes a finite series.
**Question**: Let $G$ be a group and $H$ a subgroup of $\text{Aut}(G)$ consisting of unipotent automorphisms. Is $H$ nilpotent? Moreover does $H$ stabilize a finite series for $G$?

**Remark** The question whether $H$ is nilpotent is open when $H$ is finite. (Casolo and Puglisi (2013) have though shown that $H$ is nilpotent whenever $H$ is finite and $G$ is locally graded.

**Theorem** (Casolo & Puglisi 2013). If $G$ satisfies max and $H$ is a subgroup consisting of unipotent automorphisms, then $H$ stabilizes a finite series.

**Theorem** (Frati 2014). If $G$ is abelian-by-polycyclic and $H$ is a finitely generated, then $H$ stabilizes a finite series of $G$. 
**Question**: Let $G$ be a group and $H$ a subgroup of $\text{Aut}(G)$ consisting of unipotent automorphisms. Is $H$ nilpotent? Moreover does $H$ stabilize a finite series for $G$?

**Remark** The question whether $H$ is nilpotent is open when $H$ is finite. (Casolo and Puglisi (2013) have though shown that $H$ is nilpotent whenever $H$ is finite and $G$ is locally graded.

**Theorem** (Casolo & Puglisi 2013). If $G$ satisfies max and $H$ is a subgroup consisting of unipotent automorphisms, then $H$ stabilizes a finite series.

**Theorem** (Frati 2014). If $G$ is abelian-by-polycyclic and $H$ is a finitely generated, then $H$ stabilizes a finite series of $G$.

**Theorem** (Frati 2014). If $G$ is solvable of finite Prüfer rank and $H$ is finitely generated, then $H$ stablizes a finite series of $G$. 
2. Solvable groups acting $n$-unipotently on solvable groups

Theorem (Puglisi, & T, 2017). Let $G$ be a solvable group and $H$ a solvable subgroup of $\text{Aut}(G)$ whose elements are $n$-unipotent.

(1) If $H$ is finitely generated then it stabilizes a finite series in $G$.

(2) If $G$ has a characteristic series with torsion-free factors, then $H$ stabilizes a finite series in $G$.

Remark. In particular we have that (2) holds when $G$ is nilpotent and torsion-free.

Remark. In fact one can deduce the following stronger variant of the one given in last remark. If $P_n$ is the set of prime divisors of $e(n)$ in the following result then it suffices that $G$ is $P_n$-torsion free.
2. Solvable groups acting $n$-unipotently on solvable groups

Theorem (Puglisi, & T 2017). Let $G$ be a solvable group and $H$ a solvable subgroup of Aut ($G$) whose elements are $n$-unipotent.

(1) If $H$ is finitely generated then it stabilizes a finite series in $G$.

(2) If $G$ has a characteristic series with torsion-free factors, then $H$ stabilizes a finite series in $G$. 

Remark. In particular we have that (2) holds when $G$ is nilpotent and torsion-free.

Remark. In fact one can deduce the following stronger variant of the one given in the last remark. If $P_n$ is the set of prime divisors of $e(n)$ in the following result then it suffices that $G$ is $P_n$-torsion free.
2. Solvable groups acting \( n \)-unipotently on solvable groups

**Theorem** (Puglisi, & T 2017). Let \( G \) be a solvable group and \( H \) a solvable subgroup of \( \text{Aut}(G) \) whose elements are \( n \)-unipotent.

(1) If \( H \) is finitely generated then it stabilizes a finite series in \( G \).

(2) If \( G \) has a characteristic series with torsion-free factors, then \( H \) stabilizes a finite series in \( G \).

**Remark.** In particular we have that (2) holds when \( G \) is nilpotent and torsion-free.
2. Solvable groups acting $n$-unipotently on solvable groups

**Theorem** (Puglisi, & T 2017). Let $G$ be a solvable group and $H$ a solvable subgroup of $\text{Aut}(G)$ whose elements are $n$-unipotent.

(1) If $H$ is finitely generated then it stabilizes a finite series in $G$.

(2) If $G$ has a characteristic series with torsion-free factors, then $H$ stabilizes a finite series in $G$.

**Remark.** In particular we have that (2) holds when $G$ is nilpotent and torsion-free.

**Remark.** In fact one can deduce the following stronger variant of the one given in last remark. If $\mathcal{P}_n$ is the set of prime divisors of $e(n)$ in the following result then it suffices that $G$ is $\mathcal{P}_n$-torsion free.
The proof of the results above by Frati, Casolo & Puglisi and Puglisi & T rely on the following Theorem.

\[ H^{c(n)}G = [H, c(n)G]^{e(n)}. \]
The proof of the results above by Frati, Casolo & Puglisi and Puglisi & T rely on the following Theorem.

**Theorem.** (Crosby & T 2010). Let $H$ be a normal right $n$-Engel subgroup of a group $G$ that belongs to some term of the upper central series. Then there exist positive integers $c(n), e(n)$ such that

$$ [H^{e(n)}, c(n) G] = [H, c(n) G]^{e(n)}. $$
3. Examples

Example 1. Let $G = A \wr H$ where $A$ is a cyclic group of order 2 and $H$ is a countable elementary abelian 2-group. The group $\text{Inn}(G)$ then acts $G$ as a group of 3-unipotent automorphisms but it still doesn't stabilize any finite series.

Example 2. Let $m$ be the smallest positive integer such that the Burnside variety $B(2^m)$ is not locally finite. Choose any finitely generated infinite group $G$ in $B(2^m)$. Every involution $a \in G$ induces an $(m+1)$-unipotent automorphism in $\text{Inn}(G)$ and there must exist such an involution that does not stabilize a finite series in $G$. Otherwise we get the contradiction that $G$ is finite.

Gunnar Traustason
Unipotent automorphisms of solvable groups
3. Examples

Example 1. Let $G = A \wr H$ where $A$ is a cyclic group of order 2 and $H$ is a countable elementary abelian 2-group. The group $\text{Inn}(G)$ then acts $G$ as a group of 3-unipotent automorphisms but it still doesn’t stabilize any finite series.
3. Examples

Example 1. Let $G = A \wr H$ where $A$ is a cyclic group of order 2 and $H$ is a countable elementary abelian 2-group. The group $\text{Inn}(G)$ then acts $G$ as a group of 3-unipotent automorphisms but it still doesn’t stabilize any finite series.

Example 2. Let $m$ be the smallest positive integer such that the Burnside variety $B(2^m)$ is not locally finite. Choose any finitely generated infinite group $G$ in $B(2^m)$. Every involution $a \in G$ induces an $(m+1)$-unipotent automorphism in $\text{Inn}(G)$ and there must exist such an involution that does not stabilize a finite series in $G$. Otherwise we get the contradiction that $G$ is finite.