What's new on Z_3^*?

Imke Toborg
August 6th, 2017

MLU Halle-Wittenberg

Groups St Andrews 2017 in Birmingham

Z_3^*-Theorem

Z_3^*-Theorem

Let G be a finite group and let $x \in G$ be an isolated element of order 3. Then $x \in Z_3^*(G)$.

Z_3^*-Theorem

Let G be a finite group and let $x \in G$ be an isolated element of order 3. Then $x \in Z_3^*(G)$.

Definition

Z_3^*-Theorem

Let G be a finite group and let $x \in G$ be an isolated element of order 3. Then $x \in Z_3^*(G)$.

Definition

Let G be a finite group and p be a prime.

Z_3^*-Theorem

Let G be a finite group and let $x \in G$ be an isolated element of order 3. Then $x \in Z_3^*(G)$.

Definition

Let G be a finite group and p be a prime.

- The group $O_{p'}(G)$ is the largest normal subgroup of order prime to p and $Z_p^*(G)$ is the full pre-image of $Z(G/O_{p'}(G))$.

Z_3^*-Theorem

Let G be a finite group and let $x \in G$ be an isolated element of order 3. Then $x \in Z_3^*(G)$.

Definition

Let G be a finite group and p be a prime.

- The group $O_{p'}(G)$ is the largest normal subgroup of order prime to p and $Z_p^*(G)$ is the full pre-image of $Z(G/O_{p'}(G))$.
- An element x of order p is called isolated in G if and only if for all $g \in G$ such that $x^g \in C_G(x)$, we have that $x^g = x$.

Z_3^*-Theorem

Let G be a finite group and let $x \in G$ be an isolated element of order 3. Then $x \in Z_3^*(G)$.

Definition

Let G be a finite group and p be a prime.

- The group $O_{p'}(G)$ is the largest normal subgroup of order prime to p and $Z_p^*(G)$ is the full pre-image of $Z(G/O_{p'}(G))$.
- An element x of order p is called isolated in G if and only if for all $g \in G$ such that $x^g \in C_G(x)$, we have that $x^g = x$.

Lemma

Let G be a finite group and p be a prime.
Suppose that $P \in \mathrm{Syl}_p(G)$ and that $x \in P$ has order p.
Then the following statements are equivalent:

Z_3^*-Theorem

Let G be a finite group and let $x \in G$ be an isolated element of order 3. Then $x \in Z_3^*(G)$.

Definition

Let G be a finite group and p be a prime.

- The group $O_{p'}(G)$ is the largest normal subgroup of order prime to p and $Z_p^*(G)$ is the full pre-image of $Z(G/O_{p'}(G))$.
- An element x of order p is called isolated in G if and only if for all $g \in G$ such that $x^g \in C_G(x)$, we have that $x^g = x$.

Lemma

Let G be a finite group and p be a prime.
Suppose that $P \in \mathrm{Syl}_p(G)$ and that $x \in P$ has order p.
Then the following statements are equivalent:

- $x^G \cap C_G(x) = \{x\}$.

Z_3^*-Theorem

Let G be a finite group and let $x \in G$ be an isolated element of order 3. Then $x \in Z_3^*(G)$.

Definition

Let G be a finite group and p be a prime.

- The group $O_{p'}(G)$ is the largest normal subgroup of order prime to p and $Z_p^*(G)$ is the full pre-image of $Z(G/O_{p'}(G))$.
- An element x of order p is called isolated in G if and only if for all $g \in G$ such that $x^g \in C_G(x)$, we have that $x^g = x$.

Lemma

Let G be a finite group and p be a prime.
Suppose that $P \in \mathrm{Syl}_p(G)$ and that $x \in P$ has order p.
Then the following statements are equivalent:

- $x^G \cap C_G(x) = \{x\}$.
- $x^G \cap P = \{x\}$.

Z_3^*-Theorem

Let G be a finite group and let $x \in G$ be an isolated element of order 3. Then $x \in Z_3^*(G)$.

Definition

Let G be a finite group and p be a prime.

- The group $O_{p'}(G)$ is the largest normal subgroup of order prime to p and $Z_p^*(G)$ is the full pre-image of $Z(G/O_{p'}(G))$.
- An element x of order p is called isolated in G if and only if for all $g \in G$ such that $x^g \in C_G(x)$, we have that $x^g = x$.

Lemma

Let G be a finite group and p be a prime.
Suppose that $P \in \mathrm{Syl}_p(G)$ and that $x \in P$ has order p.
Then the following statements are equivalent:

- $x^G \cap C_G(x) = \{x\}$.
- $x^G \cap P = \{x\}$.
- $x^G \cap C_P(x) = \{x\}$.

Z_3^*-Theorem

Let G be a finite group and let $x \in G$ be an isolated element of order 3. Then $x \in Z_3^*(G)$.

Definition

Let G be a finite group and p be a prime.

- The group $O_{p'}(G)$ is the largest normal subgroup of order prime to p and $Z_p^*(G)$ is the full pre-image of $Z(G/O_{p'}(G))$.
- An element x of order p is called isolated in G if and only if for all $g \in G$ such that $x^g \in C_G(x)$, we have that $x^g = x$.

Lemma

Let G be a finite group and p be a prime.
Suppose that $P \in \mathrm{Syl}_p(G)$ and that $x \in P$ has order p.
Then the following statements are equivalent:

- $x^G \cap C_G(x) = \{x\}$.
- $x^G \cap P = \{x\}$.
- $x^G \cap C_P(x) = \{x\}$.

Remark

For all $g \in G$ it is true that:

$$g \in Z(G) \Leftrightarrow g^G = \{g\}$$

Z_3^*-Theorem

Let G be a finite group and let $x \in G$ be an isolated element of order 3. Then $x \in Z_3^*(G)$.

Z_3^*-Theorem

Let G be a finite group and let $x \in G$ be an isolated element of order 3. Then $x \in Z_3^*(G)$.

Proof

Z_3^*-Theorem

Let G be a finite group and let $x \in G$ be an isolated element of order 3. Then $x \in Z_3^*(G)$.

Proof

Let G be a minimal counterexample.

Z_3^*-Theorem

Let G be a finite group and let $x \in G$ be an isolated element of order 3. Then $x \in Z_3^*(G)$.

Proof

Let G be a minimal counterexample. Then G' is simple and we have $G = G' \cdot \langle x \rangle$.

Z_3^*-Theorem

Let G be a finite group and let $x \in G$ be an isolated element of order 3. Then $x \in Z_3^*(G)$.

Proof

Let G be a minimal counterexample. Then G' is simple and we have $G = G' \cdot \langle x \rangle$. Using the Classification of the Finite Simple Groups we obtain a contradiction.

Z_3^*-Theorem

Let G be a finite group and let $x \in G$ be an isolated element of order 3. Then $x \in Z_3^*(G)$.

Proof

Let G be a minimal counterexample. Then G' is simple and we have $G = G' \cdot \langle x \rangle$. Using the Classification of the Finite Simple Groups we obtain a contradiction.

Why does the theorem hold?

Z_3^*-Theorem
Let G be a finite group and let $x \in G$ be an isolated element of order 3. Then $x \in Z_3^*(G)$.

Proof
Let G be a minimal counterexample. Then G' is simple and we have $G = G' \cdot \langle x \rangle$. Using the Classification of the Finite Simple Groups we obtain a contradiction.

Why does the theorem hold?

We would like to understand why the Z_3^*-Theorem holds.

Z_3^*-Theorem
Let G be a finite group and let $x \in G$ be an isolated element of order 3. Then $x \in Z_3^*(G)$.

Proof
Let G be a minimal counterexample. Then G' is simple and we have $G = G' \cdot \langle x \rangle$. Using the Classification of the Finite Simple Groups we obtain a contradiction.

Why does the theorem hold?

We would like to understand why the Z_3^*-Theorem holds.

The project is joint work with Rebecca Waldecker.

Why do we think it is possible to find a proof of the Z_3^*-Theorem?

Why do we think it is possible to find a proof of the Z_3^*-Theorem that is not only a proof, but provides a better understanding of group theory in general and gives structural insights?

Why do we think it is possible to find a proof of the Z_3^*-Theorem that is not only a proof, but provides a better understanding of group theory in general and gives structural insights?

- For $p = 2$ it is possible (Waldecker „Isolated involutions in finite groups." Memoirs of the AMS, Vol. 226, No. 1061 (2013).)

Why do we think it is possible to find a proof of the Z_3^*-Theorem that is not only a proof, but provides a better understanding of group theory in general and gives structural insights?

- For $p = 2$ it is possible (Waldecker „Isolated involutions in finite groups." Memoirs of the AMS, Vol. 226, No. 1061 (2013).)
- For $p = 3$ we are able to prove a weaker statement:

Why do we think it is possible to find a proof of the Z_3^*-Theorem that is not only a proof, but provides a better understanding of group theory in general and gives structural insights?

- For $p = 2$ it is possible (Waldecker „Isolated involutions in finite groups.“ Memoirs of the AMS, Vol. 226, No. 1061 (2013).)
- For $p = 3$ we are able to prove a weaker statement: Rowley's theorem on 3-locally central elements.

Why do we think it is possible to find a proof of the Z_3^*-Theorem that is not only a proof, but provides a better understanding of group theory in general and gives structural insights?

- For $p = 2$ it is possible (Waldecker „Isolated involutions in finite groups.“ Memoirs of the AMS, Vol. 226, No. 1061 (2013).)
- For $p = 3$ we are able to prove a weaker statement: Rowley's theorem on 3-locally central elements.
  - Either the local 2-structure of G is determined by the local 3-structure of G

Why do we think it is possible to find a proof of the Z_3^*-Theorem that is not only a proof, but provides a better understanding of group theory in general and gives structural insights?

- For $p = 2$ it is possible (Waldecker „Isolated involutions in finite groups.“ Memoirs of the AMS, Vol. 226, No. 1061 (2013).)
- For $p = 3$ we are able to prove a weaker statement: Rowley's theorem on 3-locally central elements.
  - Either the local 2-structure of G is determined by the local 3-structure of G
    ($\Rightarrow$ strongly embedded subgroup)

Why do we think it is possible to find a proof of the Z_3^*-Theorem that is not only a proof, but provides a better understanding of group theory in general and gives structural insights?

- For $p = 2$ it is possible (Waldecker „Isolated involutions in finite groups." Memoirs of the AMS, Vol. 226, No. 1061 (2013).)
- For $p = 3$ we are able to prove a weaker statement: Rowley's theorem on 3-locally central elements.
  - Either the local 2-structure of G is determined by the local 3-structure of G ($\Rightarrow$ strongly embedded subgroup)
  - or G looks like a $3'$-group from the 2-local point of view.

Why do we think it is possible to find a proof of the Z_3^*-Theorem that is not only a proof, but provides a better understanding of group theory in general and gives structural insights?

- For $p = 2$ it is possible (Waldecker „Isolated involutions in finite groups.“ Memoirs of the AMS, Vol. 226, No. 1061 (2013).)
- For $p = 3$ we are able to prove a weaker statement: Rowley's theorem on 3-locally central elements.
  - Either the local 2-structure of G is determined by the local 3-structure of G
    ($\Rightarrow$ strongly embedded subgroup)
  - or G looks like a $3'$-group from the 2-local point of view.
    ($\Rightarrow$ strongly closed abelian 2-subgroup)

Let G be a minimal counterexample for the Z_3^*-Theorem.

Let G be a minimal counterexample for the Z_3^*-Theorem.

More precisely, we choose $x \in G$ to be isolated in G and of order 3 such that $x \notin Z_3^*(G)$,

Let G be a minimal counterexample for the Z_3^*-Theorem.

More precisely, we choose $x \in G$ to be isolated in G and of order 3 such that $x \notin Z_3^*(G)$, but we suppose that Z_3^*-Theorem holds for every proper section of G.

Let G be a minimal counterexample for the Z_3^*-Theorem.

More precisely, we choose $x \in G$ to be isolated in G and of order 3 such that $x \notin Z_3^*(G)$, but we suppose that Z_3^*-Theorem holds for every proper section of G.

- Then we have that G' is simple and $G = G' \cdot \langle x \rangle$.

Let G be a minimal counterexample for the Z_3^*-Theorem.

More precisely, we choose $x \in G$ to be isolated in G and of order 3 such that $x \notin Z_3^*(G)$, but we suppose that Z_3^*-Theorem holds for every proper section of G.

- Then we have that G' is simple and $G = G' \cdot \langle x \rangle$.
- $C_G(x) = N_G(\langle x \rangle)$ is some interesting 3-local subgroup.

Let G be a minimal counterexample for the Z_3^*-Theorem.

More precisely, we choose $x \in G$ to be isolated in G and of order 3 such that $x \notin Z_3^*(G)$, but we suppose that Z_3^*-Theorem holds for every proper section of G.

- Then we have that G' is simple and $G = G' \cdot \langle x \rangle$.
- $C_G(x) = N_G(\langle x \rangle)$ is some interesting 3-local subgroup.
- If P is some Sylow 3-subgroup of G such that $x \in P$, then we have $P \leq C_G(x)$.

Let G be a minimal counterexample for the Z_3^*-Theorem.

More precisely, we choose $x \in G$ to be isolated in G and of order 3 such that $x \notin Z_3^*(G)$, but we suppose that Z_3^*-Theorem holds for every proper section of G.

- Then we have that G' is simple and $G = G' \cdot \langle x \rangle$.
- $C_G(x) = N_G(\langle x \rangle)$ is some interesting 3-local subgroup.
- If P is some Sylow 3-subgroup of G such that $x \in P$, then we have $P \leq C_G(x)$.
- Frobenius' p-complement Theorem implies that the Sylow 3-subgroups of G are not cyclic.

We assume that $C_G(x)$ is soluble.

We assume that $C_G(x)$ is soluble.

For $x \in H \lneq G$ we have $H = C_H(x) \cdot O_{3'}(H)$.
In particular, H is 3-soluble.

We assume that $C_G(x)$ is soluble.

For $x \in H \lneq G$ we have $H = C_H(x) \cdot O_{3'}(H)$.
In particular, H is 3-soluble.

Further assumptions on $P \in \mathrm{Syl}_3(G)$

We assume that $C_G(x)$ is soluble.

For $x \in H \lneq G$ we have $H = C_H(x) \cdot O_{3'}(H)$.
In particular, H is 3-soluble.

Further assumptions on $P \in \mathrm{Syl}_3(G)$

provide a strongly 3-embedded subgroup M of G,

We assume that $C_G(x)$ is soluble.

For $x \in H \lneq G$ we have $H = C_H(x) \cdot O_{3'}(H)$.
In particular, H is 3-soluble.

Further assumptions on $P \in \mathrm{Syl}_3(G)$

provide a strongly 3-embedded subgroup M of G, i.e. we have $N_G(R) \leq M$ for all $1 \neq R \leq P$.

We assume that $C_G(x)$ is soluble.

For $x \in H \lneq G$ we have $H = C_H(x) \cdot O_{3'}(H)$.
In particular, H is 3-soluble.

Further assumptions on $P \in \mathrm{Syl}_3(G)$

Suppose that P has an elementary abelian subgroup of order 27 and that P is connected.

provide a strongly 3-embedded subgroup M of G, i.e. we have $N_G(R) \leq M$ for all $1 \neq R \leq P$.

We assume that $C_G(x)$ is soluble.

For $x \in H \lneq G$ we have $H = C_H(x) \cdot O_{3'}(H)$.
In particular, H is 3-soluble.

Further assumptions on $P \in \mathrm{Syl}_3(G)$

Suppose that P has an elementary abelian subgroup of order 27 and that P is connected.

provide a strongly 3-embedded subgroup M of G, i.e. we have $N_G(R) \leq M$ for all $1 \neq R \leq P$.

- We have $x \in C_G(x) \leq M$ and M is a maximal subgroup of G.

We assume that $C_G(x)$ is soluble.

For $x \in H \lneq G$ we have $H = C_H(x) \cdot O_{3'}(H)$.
In particular, H is 3-soluble.

Further assumptions on $P \in \mathrm{Syl}_3(G)$

Suppose that P has an elementary abelian subgroup of order 27 and that P is connected.

provide a strongly 3-embedded subgroup M of G, i.e. we have $N_G(R) \leq M$ for all $1 \neq R \leq P$.

- We have $x \in C_G(x) \leq M$ and M is a maximal subgroup of G. In particular, M is 3-soluble.

We assume that $C_G(x)$ is soluble.

For $x \in H \lneq G$ we have $H = C_H(x) \cdot O_{3'}(H)$.
In particular, H is 3-soluble.

Further assumptions on $P \in \mathrm{Syl}_3(G)$

Suppose that P has an elementary abelian subgroup of order 27 and that P is connected.

provide a strongly 3-embedded subgroup M of G, i.e. we have $N_G(R) \leq M$ for all $1 \neq R \leq P$.

- We have $x \in C_G(x) \leq M$ and M is a maximal subgroup of G. In particular, M is 3-soluble.
- If H is some 3-soluble subgroup of G and if $P \cap H$ is not cyclic, then $H \leq M$.

We assume that $C_G(x)$ is soluble.

For $x \in H \lneq G$ we have $H = C_H(x) \cdot O_{3'}(H)$.
In particular, H is 3-soluble.

Further assumptions on $P \in \mathrm{Syl}_3(G)$

Suppose that P has an elementary abelian subgroup of order 27 and that P is connected.

provide a strongly 3-embedded subgroup M of G, i.e. we have $N_G(R) \leq M$ for all $1 \neq R \leq P$.

- We have $x \in C_G(x) \leq M$ and M is a maximal subgroup of G. In particular, M is 3-soluble.
- If H is some 3-soluble subgroup of G and if $P \cap H$ is not cyclic, then $H \leq M$.
- For all $q \in \pi(M)$ there is some $Q \in \mathrm{Syl}_q(M)$ such that we have $QP = PQ$ and $x \in N_G(Q)$.

We suppose for all involutions a of G that $C_G(a)$ is 3-soluble.

We suppose for all involutions a of G that $C_G(a)$ is 3-soluble.

- Then $O(C_G(a)) = 1$ for all involutions $a \in G$.

We suppose for all involutions a of G that $C_G(a)$ is ~~3-~~soluble.

- Then $O(C_G(a)) = 1$ for all involutions $a \in G$.
- G has local charakteristic 2,

We suppose for all involutions a of G that $C_G(a)$ is ~~3'-~~soluble.

- Then $O(C_G(a)) = 1$ for all involutions $a \in G$.
- G has local charakteristic 2, i.e. $F^*(N_G(A)) = O_2(N_G(A))$ for all 2-subgroups $1 \neq A$ of G.

We suppose for all involutions a of G that $C_G(a)$ is ~~3'-~~soluble.

- Then $O(C_G(a)) = 1$ for all involutions $a \in G$.
- G has local charakteristic 2, i.e. $F^*(N_G(A)) = O_2(N_G(A))$ for all 2-subgroups $1 \neq A$ of G.
- M is a 2-local subgroup of G.

We suppose for all involutions a of G that $C_G(a)$ is ~~3-~~soluble.

- Then $O(C_G(a)) = 1$ for all involutions $a \in G$.
- G has local charakteristic 2, i.e. $F^*(N_G(A)) = O_2(N_G(A))$ for all 2-subgroups $1 \neq A$ of G.
- M is a 2-local subgroup of G.
- If T is some x-invariant Sylow 2-subgroup of M, then M is the unique maximal subgroup of G that contains $\langle x \rangle T$.

We suppose for all involutions a of G that $C_G(a)$ is ~~3'-~~soluble.

- Then $O(C_G(a)) = 1$ for all involutions $a \in G$.
- G has local charakteristic 2, i.e. $F^*(N_G(A)) = O_2(N_G(A))$ for all 2-subgroups $1 \neq A$ of G.
- M is a 2-local subgroup of G.
- If T is some x-invariant Sylow 2-subgroup of M, then M is the unique maximal subgroup of G that contains $\langle x \rangle T$.

THEOREM B.7.8 (Global $C(G,T)$-Theorem). *Let G be a finite simple group of characteristic 2 type with $T \in Syl_2(G)$. Suppose that $C(G,T) < G$. Then one of the following conclusions holds:*

(1) $G \cong A_6$, $L_3(2)$, $L_3(3)$, M_{11}, or $L_2(p)$ for p a Fermat or Mersenne prime;
(2) $G \cong M_{22}$ or M_{23}; or
(3) $G \cong L_2(2^n)$, $Sz(2^n)$, $U_3(2^n)$, $L_3(2^n)$, or $Sp(4,2^n)$, with $n \geq 2$.

M. Aschbacher, R. Lyons, S. D. Smith, R. Solomon. *The Classification of Finite Simple Groups. Groups of Characteristic 2 Type*. Mathematical Surveys and Monographs, 172. American Mathematical Society, Providence, RI, 2011. p. 296.

Theorem

Theorem

Let G be a finite group and let $x \in G$ be an isolated element of order 3.

Then $x \in Z_3^*(G)$.

Theorem

Let G be a finite group and let $x \in G$ be an isolated element of order 3.
Suppose that the Z_3^*-Theorem holds in every proper section of G.
We assume that $C_G(x)$ is soluble and that for all involutions a of G that $C_G(a)$ is soluble.
Moreover, let G contain some elementary abelian subgroup of order 27 and let G be connected for the prime 3.

Then $x \in Z_3^*(G)$.

Theorem

Let G be a finite group and let $x \in G$ be an isolated element of order 3.
Suppose that the Z_3^*-Theorem holds in every proper section of G.
We assume that $C_G(x)$ is soluble and that for all involutions a of G that $C_G(a)$ is soluble.
Moreover, let G contain some elementary abelian subgroup of order 27 and let G be connected for the prime 3.

Then $x \in Z_3^*(G)$.

Let G be a finite group and let $x \in G$ be an isolated element of order 3.
Suppose that the Z_3^*-Theorem holds in every proper section of G.

We assume that $C_G(x)$ is soluble and that for all involutions a of G that $C_G(a)$ is $3-$soluble.
Moreover, let G contain some elementary abelian subgroup of order 27 and let G be connected for the prime 3.

Then $x \in Z_3^*(G)$.

Let G be a finite group and let $x \in G$ be an isolated element of order 3.
Suppose that the Z_3^*-Theorem holds in every proper section of G.
We assume that $C_G(x)$ is soluble and that for all involutions a of G that $C_G(a)$ is $3-$soluble.
Moreover, let G contain some elementary abelian subgroup of order 27 and let G be connected for the prime 3.

Then $x \in Z_3^*(G)$.

Let G be a finite group and let $x \in G$ be an isolated element of order 3.
Suppose that the Z_3^*-Theorem holds in every proper section of G.
We assume that $C_G(x)$ is soluble and that for all involutions a of G that $C_G(a)$ is $3-$soluble.
Moreover, let G contain some elementary abelian subgroup of order 27 ~~and let G be connected for the prime 3~~.

Then $x \in Z_3^*(G)$.

Let G be a finite group and let $x \in G$ be an isolated element of order 3.
Suppose that the Z_3^*-Theorem holds in every proper section of G.
We assume that $C_G(x)$ is soluble and that for all involutions a of G that $C_G(a)$ is $3-$soluble.
Moreover, let G contain some elementary abelian subgroup of order 27 ~~and let G be connected for the prime 3~~.

Then $x \in Z_3^*(G)$.

Let G be a finite group and let $x \in G$ be an isolated element of order 3.
Suppose that the Z_3^*-Theorem holds in every proper section of G.
We assume that $C_G(x)$ is $3-$soluble and that for all involutions a of G that $C_G(a)$ is $3-$soluble.
Moreover, let G contain some elementary abelian subgroup of order 27 ~~and let G be connected for the prime 3~~.

Then $x \in Z_3^*(G)$.

Let G be a finite group and let $x \in G$ be an isolated element of order 3.
Suppose that the Z_3^*-Theorem holds in every proper section of G.

We assume that $C_G(x)$ is $3-$soluble and that for all involutions a of G that $C_G(a)$ is $3-$soluble.
Moreover, let G contain some elementary abelian subgroup of order 27 ~~and let G be connected for the prime 3~~.

Then $x \in Z_3^*(G)$.

Let G be a finite group and let $x \in G$ be an isolated element of order 3.
Suppose that the Z_3^*-Theorem holds in every proper section of G.
We assume that $C_G(x)$ is $3-$soluble and that for some involutions a of G that $C_G(a)$ is $3-$soluble.
Moreover, let G contain some elementary abelian subgroup of order 27 ~~and let G be connected for the prime 3~~.

Then $x \in Z_3^*(G)$.

Thank you for
your attention!