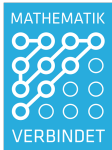


# What's new on $Z_3^*$ ?



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August 6th, 2017

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Groups St Andrews 2017 in  
Birmingham

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### Remark

For all  $g \in G$  it is true that:

$$g \in Z(G) \Leftrightarrow g^G = \{g\}$$

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The project is joint work with [Rebecca Waldecker](#).

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*THEOREM B.7.8 (Global  $C(G, T)$ -Theorem). Let  $G$  be a finite simple group of characteristic 2 type with  $T \in \text{Syl}_2(G)$ . Suppose that  $C(G, T) < G$ . Then one of the following conclusions holds:*

- (1)  $G \cong A_6, L_3(2), L_3(3), M_{11},$  or  $L_2(p)$  for  $p$  a Fermat or Mersenne prime;*
- (2)  $G \cong M_{22}$  or  $M_{23}$ ; or*
- (3)  $G \cong L_2(2^n), Sz(2^n), U_3(2^n), L_3(2^n),$  or  $Sp(4, 2^n),$  with  $n \geq 2$ .*

M. Aschbacher, R. Lyons, S. D. Smith, R. Solomon. *The Classification of Finite Simple Groups. Groups of Characteristic 2 Type*. Mathematical Surveys and Monographs, 172. American Mathematical Society, Providence, RI, 2011. p. 296.

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Suppose that the  $Z_3^*$ -Theorem holds in every proper section of  $G$ .

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Moreover, let  $G$  contain some elementary abelian subgroup of order 27 and let  $G$  be connected for the prime 3.

Then  $x \in Z_3^*(G)$ .



## Theorem

Let  $G$  be a finite group and let  $x \in G$  be an isolated element of order 3.

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Thank you for  
your attention!

