Cayley-automatic groups and semigroups

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Notation

A : a finite set of symbols.

A^* : the set of all (finite) words formed from the symbols in A (including the *empty word* ε).

If we take non-empty words (i.e. if we omit ε) then we get A^+.

A^+ is a semigroup (under concatenation).

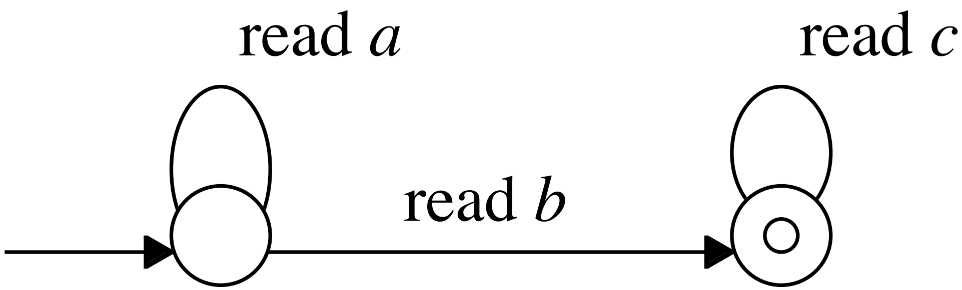
A^* is a monoid with identity ε.

If M is a monoid (respectively S is a semigroup) generated by a finite set A then there is a natural homomorphism $\varphi : A^* \to M$ (respectively $\varphi : A^+ \to S$).

A *language* is a subset of A^* (for some finite set A).

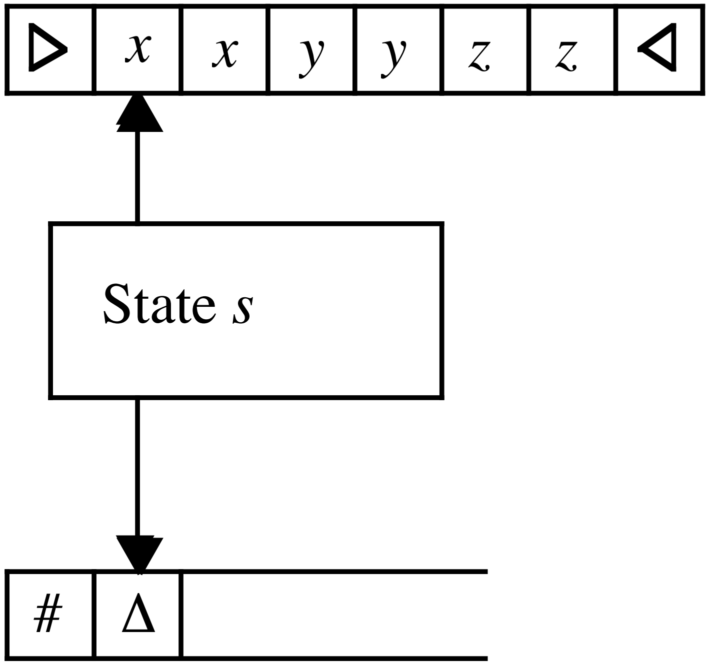
Regular languages are the languages accepted by *finite automata*.

A word α is *accepted* by an automaton M if α maps the start state to an accept state. For example, the finite automaton below accepts the language $\{a^n b c^m : n, m \in \mathbf{N}\}$:



Allowing nondeterminism here does not increase the set of languages accepted.

We can also consider a general model of computation such as a *Turing machine*.



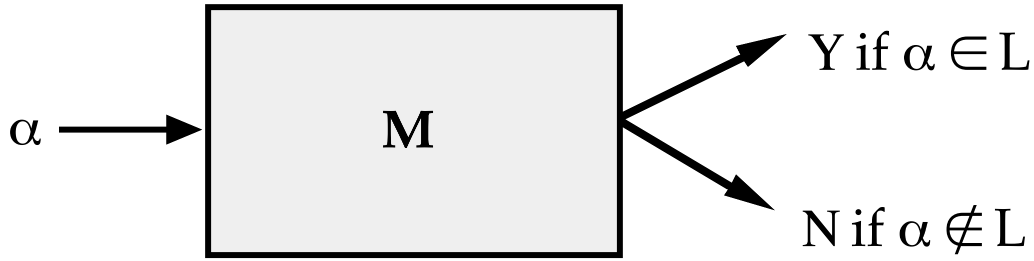
Here we have some memory (in the form of a “work tape”) as well as the input.

A Turing machine with a given input will either

(i) terminate (if it enters a halt state); or

(ii) hang (no legal move defined); or

(iii) run indefinitely without terminating.



We will take a *decision-making Turing machine* (one that always terminates and outputs true or false) here (we are considering the class of *recursive languages*).

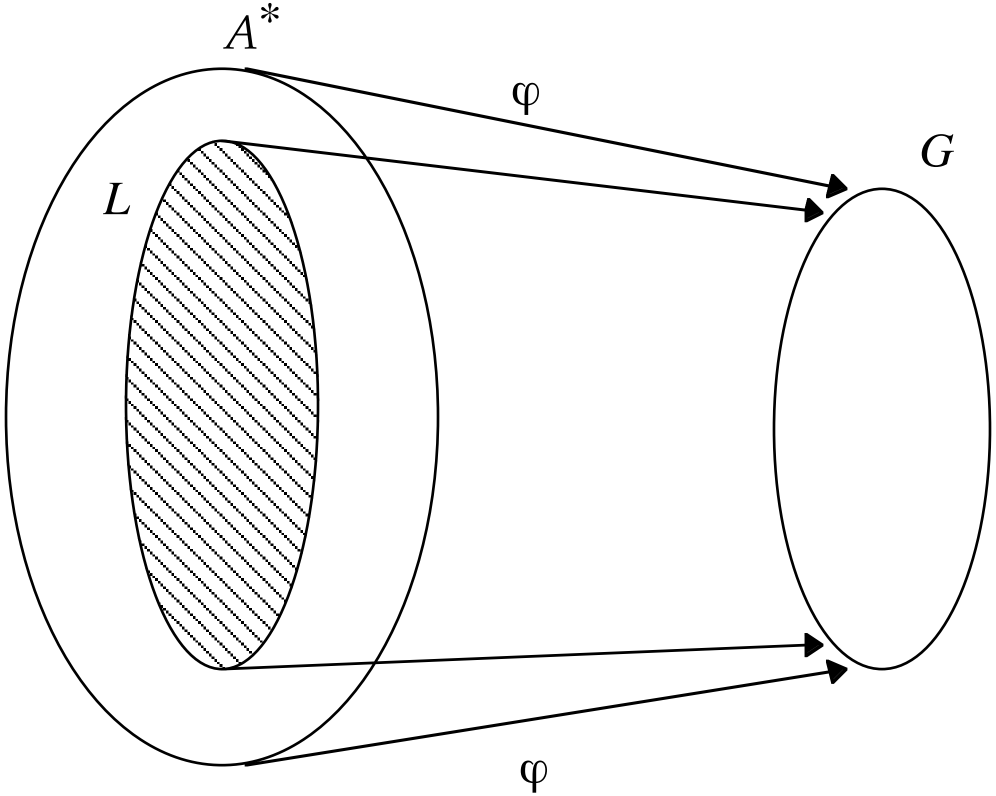
A *structure* $S = (D, R_1, R_2, \ldots, R_n)$ consists of:

- a set D, called the *domain* of S;
- relations $R_1, R_2, \ldots, R_n$ such that, for each i with $1 \leq i \leq n$, there exists $r = r_i \geq 1$ with R_i a subset of D^r; r is called the *arity* of the relation R_i.

A structure $S = (D, R_1, R_2, \ldots, R_n)$ is said to be *computable* if:

- there is a set of symbols A such that $D \subseteq A^*$ and there is a decision-making Turing machine for D;
- for each R_i of arity r there is a decision-making Turing machine that, on input $(a_1, a_2, \ldots, a_r)$, outputs *true* if $a_i \in D$ for each i and if $(a_1, a_2, \ldots, a_r) \in R_i$ and outputs *false* otherwise.

Automatic groups



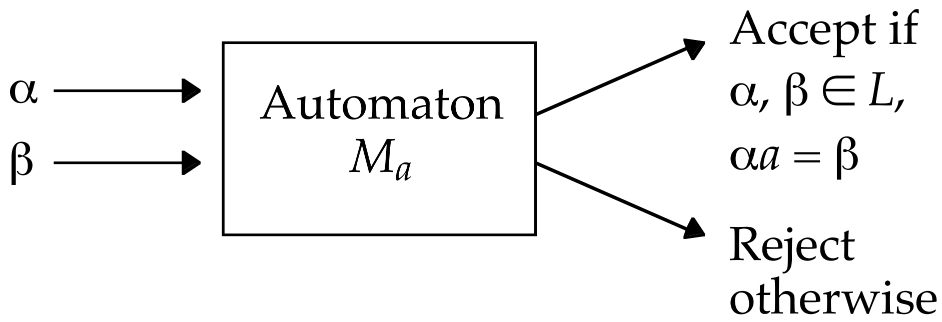
L is a regular subset of A^* (or A^+). The general idea is that "multiplication in the group G is recognized by automata".

When we talk about "accepting" a pair (or, more generally, a tuple) of words, we are "padding" the shorter words with a new symbol (say \$) to make the words all the same length:

a_1	a_2	a_3	…………	a_n	\$	………	\$
b_1	b_2	b_3	…………	b_n	b_{n+1}	………	b_m

We are thus reading the different words "synchronously".

For automatic groups, for each $a \in A$, there is a finite automaton M_a such that



Automaticity generalizes naturally to semigroups (but not to other structures in an obvious way).

Another notion called *FA-presentability* was introduced by B. Khoussainov & A. Nerode; this applies to general structures.

A structure $\mathcal{S} = (D, R_1, R_2, \ldots, R_n)$ is said to be *FA-presentable* if:

- there is a regular language L and a bijective map $\varphi : L \to D$;
- for each relation R_i of arity r, there is a finite automaton that accepts a tuple $(a_1, a_2, \ldots, a_r)$ if and only if $a_p \in L$ for all p and $(a_1, a_2, \ldots, a_r) \in R_i$.

> If $\mathcal{S}$ is an FA-presentable structure then the first-order theory of $\mathcal{S}$ is decidable. B. Khoussainov & A. Nerode

An ordinal α is FA-presentable if and only if $\alpha < \omega^{\omega}$. C. Delhommé

An integral domain is FA-presentable if and only if it is finite.

B. Khoussainov, A. Nies, S. Rubin & F. Stephan

An infinite Boolean algebra is FA-presentable if and only if it of the form $\mathcal{B}^n$ (some $n \in \mathbf{N}$), where $\mathcal{B}$ is the Boolean algebra of finite and cofinite subsets of $\mathbf{N}$. B. Khoussainov, A. Nies, S. Rubin & F. Stephan

A fin gen group is FA-presentable if and only if it is virtually abelian. G. P. Oliver & R. M. Thomas

Consequence: if a fin gen group is FA-presentable then it is automatic (but the converse is false). What about semigroups?

A fin gen commutative semigroup:

- need not be automatic. M. Hoffmann & R. M. Thomas
- is FA-presentable. A. J. Cain, N. Ruskuc, G. P. Oliver & R. M. Thomas

So a fin gen FA-presentable semigroup need not be automatic.

A fin gen cancellative semigroup is FA-presentable if and only if it embeds in a (fin gen) virtually abelian group.

A. J. Cain, N. Ruskuc, G. P. Oliver & R. M. Thomas

There is a fin gen non-automatic semigroup that is a subsemigroup of a virtually abelian group; so a fin gen cancellative FA-presentable semigroup need not be automatic. A. J. Cain

Given a group G with a finite set of generators $A = \{a_1, \ldots, a_n\}$, we form a new structure $\mathcal{G} = (G, R_1, \ldots, R_n)$ where $(g, h) \in R_i$ if and only if $ga_i = h$; this is called the *Cayley graph* of G with respect to A.

If G is an automatic group then we have an encoding of the elements of G as words in A^* such that there are finite automata recognizing multiplication by elements of A. So, if G is an automatic group then the Cayley graph $\mathcal{G}$ is FA-presentable (but the converse is false).

G fin gen FA-presentable $\Rightarrow$ G automatic $\Rightarrow$ $\mathcal{G}$ FA-presentable

We say that a fin gen group G is *CGA* (*Cayley graph automatic*) if its Cayley graph $\mathcal{G}$ is FA-presentable. This generalizes naturally to fin gen semigroups.

S fin gen FA-presentable $\Rightarrow$ S CGA

S automatic $\Rightarrow$ S CGA

If G is a CGA group then the word problem for G can be solved in quadratic time. O. Kharlampovich, B. Khoussainov & A. Miasnikov

This result generalizes to CGA semigroups.

A. J. Cain, R. Carey, N. Ruskuc & R. M. Thomas

Cayley graph automaticity for groups is preserved under:

- finite extensions;
- direct products;
- free products;
- fin gen regular subgroups;
- certain semidirect products;
- certain amalgamated free products;

O. Kharlampovich, B. Khoussainov & A. Miasnikov

- wreath products with **Z**; D. Berdinsky & B. Khoussainov

So CGA groups are not necessarily finitely presented.

Fin gen nilpotent groups of class at most 2 are CGA.

O. Kharlampovich, B. Khoussainov & A. Miasnikov

Baumslag-Solitar groups $< a, t : t^{-1}a^m t = a^n >$ are CGA.

D. Berdinsky & B. Khoussainov

The conjugacy problem is undecidable for CGA groups.

The isomorphism problem is undecidable for CGA groups.

A. Miasnikov & Z Sunic

CGA semigroups. Joint work with A. J. Cain, R. Carey & N. Ruskuc

Cayley graph automaticity for semigroups is preserved under:

- subsemigroups of finite Rees index;
- fin gen regular subsemigroups;
- direct products (if the product is fin gen);
- certain semidirect products;
- fin gen Rees matrix semigroups.
- zero unions;
- free products;

There are some complete classifications (for example, when a strong semilattice of semigroups is a CGA semigroup).

Many open questions here – work in progress!

A structure $S = (D, R_1, \ldots, R_n)$ is said to be unary *FA-presentable* if:

- there is a regular language L over an alphabet consisting of one symbol and a bijective map $\varphi : L \to D$;
- for each relation R_i of arity r, there is a finite automaton that accepts a tuple $(a_1, a_2, \ldots, a_r)$ if and only if $a_p \in L$ for all p and $(a_1, a_2, \ldots, a_r) \in R_i$.

Which structures are unary FA-presentable?

Cancellative unary FA-presentable semigroups are finite.

(This generalizes a previous result for groups by A. Blumensath.)

Fin gen unary FA-presentable semigroups are finite.

(In general, unary FA-presentable semigroups are locally finite.)

A. J. Cain, N, Ruskuc & R. M. Thomas

What about unary CGA semigroups?

A cancellative semigroup is unary CGA if and only if it embeds into a virtually cyclic group.

A. J. Cain, R. Carey, N. Ruskuc & R. M. Thomas

Thank you!