On some of the subgroups of $\mathrm{E}_6(q)$ and ${}^2\mathrm{E}_6(q)$.

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Split octonions

If F is any field, the split-octonion algebra over F is an 8-dimensional vector space $\mathbb{O} = \mathbb{O}_F$ over F, with basis $\{e_i \mid i \in \pm I\}$, where $I = \{0, 1, \omega, \overline{\omega}\}$ and bilinear multiplication given by the following table.

	e_{-1}	$e_{\overline{\omega}}$	e_{ω}	e_0	e_{-0}	$e_{-\omega}$	$e_{-\overline{\omega}}$	e_1
e_{-1}	0	0	0	0	e_{-1}	$e_{\overline{\omega}}$	$-e_{\omega}$	$-e_0$
$e_{\overline{\omega}}$	0	0	$-e_{-1}$	$e_{\overline{\omega}}$	0	0	$-e_{-0}$	$e_{-\omega}$
e_{ω}	0	e_{-1}	0	e_{ω}	0	$-e_{-0}$	0	$-e_{-\overline{\omega}}$
e_0	e_{-1}	0	0	e_0	0	$e_{-\omega}$	$e_{-\overline{\omega}}$	0
e_{-0}	0	$e_{\overline{\omega}}$	e_{ω}	0	e_{-0}	0	0	e_1
$e_{-\omega}$	$-e_{\overline{\omega}}$	0	$-e_0$	0	$e_{-\omega}$	0	e_1	0
$e_{-\overline{\omega}}$	e_{ω}	$-e_0$	0	0	$e_{-\overline{\omega}}$	$-e_1$	0	0
e_1	$-e_{-0}$	$-e_{-\omega}$	$e_{-\overline{\omega}}$	e_1	0	0	0	0

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The norm $N(x) = x\overline{x}$ can be computed as

$$N(\sum_{i \in \pm I} \lambda_i e_i) = \sum_{i \in I} \lambda_i \lambda_{-i}.$$

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Lemma (Moufang law)

For all $x, y, z \in \mathbb{O}$, the following identities hold:

$$\begin{aligned} x(yz)x &= (xy)(zx), \\ x(yzy) &= ((xy)z)x, \\ (xyx)z &= x(y(xz)). \end{aligned}$$

Albert set

Consider the set $\mathbb{J} = \mathbb{J}_F$ of 3×3 Hermitean matrices written over the octonions. For an element $X \in \mathbb{J}$ we write

$$X = (a, b, c \mid A, B, C) = \begin{pmatrix} a & C & \overline{B} \\ \overline{C} & b & A \\ B & \overline{A} & c \end{pmatrix}.$$

Albert set

Cayley-Hamilton Theorem

Any matrix $X = (a, b, c \mid A, B, C)$ in $\mathbb{J}$ satisfies

$$X^3 = \mathrm{Tr}(X).X^2 + Q(X).X + \det(X).I,$$

where

$$\begin{aligned} Q(X) &= A\overline{A} + B\overline{B} + C\overline{C} - ab - ac - bc, \\ \det(X) &= abc - aA\overline{A} - bB\overline{B} - cC\overline{C} + (AB)C + \overline{C}(\overline{BA}). \end{aligned}$$

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Corollary

Any automorphism of the algebra preserves the trace $\mathrm{Tr}(X)$, the standard norm $N(X) = \mathrm{Tr}(X^2)$, and the determinant. Moreover,

$$\det(X) = \frac{1}{3}\mathrm{Tr}(X^3) - \frac{1}{2}\mathrm{Tr}(X^2)\mathrm{Tr}(X) + \frac{1}{6}\mathrm{Tr}(X)^3.$$

Action of the automorphism group

If M is any 3×3 matrix written over any "complex" subalgebra of the octonions, then the operation $X \mapsto \overline{M}^T XM$ makes sense, because each entry in $\overline{M}^T XM$ is a sum of terms of the form $m_1 x m_2$, and $(m_1 x)m_2 = m_1(x m_2)$.

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Suppose now $F = \mathbb{F}_q$. The group $\mathrm{SE}_6(q)$ is defined as the group of linear maps which preserve the determinant.

Define the Hermitean form H on $\mathbb{J}_{\mathbb{F}_{q^2}}$ by

$$H(a, b, c \mid A, B, C) = aa' + bb' + cc' + \mathrm{Tr}(A\overline{A}' + B\overline{B}' + C\overline{C}'),$$

where $'$ is an automorphism of $\mathbb{O}$ induced by the field automorphism $\lambda \mapsto \lambda^q$.

The group ${}^2\mathrm{SE}_6(q)$

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where $'$ is an automorphism of $\mathbb{O}$ induced by the field automorphism $\lambda \mapsto \lambda^q$.

The group ${}^2\mathrm{SE}_6(q)$ is defined as the subgroup of $\mathrm{SE}_6(q^2)$ which preserves H.

Some elements of $\mathrm{SE}_6(q)$

Coordinate permitations, generated by

$$\begin{array}{rcl} (a, b, c \mid A, B, C) & \mapsto & (c, a, b \mid C, A, B), \\ (a, b, c \mid A, B, C) & \mapsto & (a, c, b \mid \overline{A}, \overline{C}, \overline{B}) \end{array}$$

preserve the determinant. These are encoded by

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

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The group contains the diagonal matrices

$$P_u = \begin{pmatrix} u & 0 & 0 \\ 0 & \overline{u} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $u\overline{u} = 1$.

$\mathrm{Spin}_8^+(q)$

The action of $P_u = \mathrm{diag}(u, \overline{u}, 1)$ on the Albert set is given by

$$(a, b, c \mid A, B, C) \mapsto (a, b, c \mid uA, Bu, \overline{u}C\overline{u}).$$

Essentially, the map $C \mapsto \overline{u}C\overline{u}$ is a product of two reflections and the action generated is that of group $\mathrm{Spin}_8^+(q)$.

$\mathrm{Spin}_3(q)$

Consider the matrices

$$Q_{s,t} = \begin{pmatrix} s & t & 0 \\ -t^q & s^q & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $s, t \in \mathbb{F}_{q^2}$ are such that $s^{q+1} + t^{q+1} = 1$. The totality of these matrices is $q(q^2 - 1)$ and in fact these form a group $\mathrm{Spin}_3(q) \cong \mathrm{SU}_2(q)$.

$\mathrm{Spin}_{10}^{-}(q)$

Consider the group generated by the matrices P_u and $Q_{s,t}$. Denote by V_{10}^{-} the subspace $(\lambda, -\lambda^q, 0 \mid 0, 0, C)$ where $\lambda \in \mathbb{F}_{q^2}$ and C is an octonion written over $\mathbb{F}_q$. Let Q_{10}^{-} be the quadratic form defined on V_{10}^{-} by

$$Q_{10}^{-}((\lambda, -\lambda^q, 0 \mid 0, 0, C)) = \lambda\lambda^q + C\overline{C}.$$

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We notice that Q_{10}^{-} is of *minus* type.

The elements P_u and $Q_{s,t}$ preserve Q_{10}^{-} and in fact these generate a group $\mathrm{Spin}_{10}^{-}(q)$, except for $q = 2$.

Stabilising $(0,0,0 \mid 0,0,e_1)$.

Denote $X = (0,0,0 \mid 0,0,e_1)$ and consider the subgroup $\mathrm{Stab}_G(X)$, where $G = {}^2\mathrm{SE}_6(q)$. In fact, the matrices P_u with the additional constraint $u_{-1} = u_1 = 0$ do the job as well as the matrices $Q_{s,t}$. Such matrices P_u generate $\mathrm{Spin}_6^+(q)$ and combining that with $Q_{s,t}$ we get $\mathrm{Spin}_8^-(q)$. But there is more.

Consider the matrices M_x and L_x defined by

$$M_x = \begin{pmatrix} 1 & 0 & -\overline{x}' \\ 0 & 1 & 0 \\ x & 0 & 1 \end{pmatrix}, \quad L_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & -\overline{x}' & 1 \end{pmatrix}.$$

These also stabilise X and generate a subgroup of shape q^{8+16}.

Overall we get a subgroup of shape $q^{8+16}.\mathrm{Spin}_8^-(q)$.

Thank you for your attention!