On some of the subgroups of $E_6(q)$ and $^2E_6(q)$.

Yegor Stepanov

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If $F$ is any field, the split-octonion algebra over $F$ is an 8-dimensional vector space $\mathbb{O} = \mathbb{O}_F$ over $F$, with basis \( \{e_i \mid i \in \pm I\} \), where $I = \{0, 1, \omega, \bar{\omega}\}$ and bilinear multiplication given by the following table.

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If $x \in \mathbb{O}$, then $x = \sum_{i \in \pm I} \lambda_i e_i$, where $I = \{0, 1, \omega, \overline{\omega}\}$. 
If $x \in \mathbb{O}$, then $x = \sum_{i \in \pm l} \lambda_i e_i$, where $l = \{0, 1, \omega, \overline{\omega}\}$.

$\text{Tr}(\sum_{i \in \pm l} \lambda_i e_i) = \lambda_0 + \lambda_{-0}$. 
If $x \in O$, then $x = \sum_{i \in \pm I} \lambda_i e_i$, where $I = \{0, 1, \omega, \overline{\omega}\}$.

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The anti-automorphism $x \mapsto \overline{x}$ takes the form

$e_0 \mapsto e_{-0}, \quad e_i \mapsto -e_i, \quad i \neq \pm 0$. 

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Split octonions

If $x \in \mathbb{O}$, then $x = \sum_{i \in \pm I} \lambda_i e_i$, where $I = \{0, 1, \omega, \overline{\omega}\}$.

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The anti-automorphism $x \mapsto \overline{x}$ takes the form

$e_0 \mapsto e_{-0}, \ e_i \mapsto -e_i, \ i \neq \pm 0$.

The norm $N(x) = x\overline{x}$ can be computed as

$N\left( \sum_{i \in \pm I} \lambda_i e_i \right) = \sum_{i \in I} \lambda_i \lambda_{-i}$.
Lemma
If $x, y, z \in \mathcal{O}$, then

$$\text{Tr}(x(yz)) = \text{Tr}((xy)z).$$
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Split octonions

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**Lemma (Moufang law)**
For all \( x, y, z \in \mathbb{O} \), the following identities hold:
\[
\begin{align*}
x(yz)x &= (xy)(zx), \\
x(yzy) &= ((xy)z)x, \\
(xy)xz &= x(y(xz)).
\end{align*}
\]
Consider the set $\mathbb{J} = \mathbb{J}_F$ of $3 \times 3$ Hermitean matrices written over the octonions. For an element $X \in \mathbb{J}$ we write

$$X = (a, b, c \mid A, B, C) = \begin{pmatrix} a & C & \bar{B} \\ \bar{C} & b & A \\ B & \bar{A} & c \end{pmatrix}.$$
Albert set

Cayley-Hamilton Theorem

Any matrix \( X = (a, b, c \mid A, B, C) \) in \( \mathbb{J} \) satisfies

\[
X^3 = \text{Tr}(X)X^2 + Q(X)X + \det(X)I,
\]

where

\[
Q(X) = A\overline{A} + B\overline{B} + C\overline{C} - ab - ac - bc,
\]

\[
\det(X) = abc - aA\overline{A} - bB\overline{B} - cC\overline{C} + (AB)C + \overline{C}(BA).
\]
Cayley-Hamilton Theorem

Any matrix $X = (a, b, c | A, B, C)$ in $\mathbb{J}$ satisfies

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$$\text{det}(X) = abc - aA\overline{A} - bB\overline{B} - cC\overline{C} + (AB)C + \overline{C}(BA).$$

Corollary

Any automorphism of the algebra preserves the trace $\text{Tr}(X)$, the standard norm $N(X) = \text{Tr}(X^2)$, and the determinant. Moreover,

$$\text{det}(X) = \frac{1}{3} \text{Tr}(X^3) - \frac{1}{2} \text{Tr}(X^2)\text{Tr}(X) + \frac{1}{6} \text{Tr}(X)^3.$$
If $M$ is any $3 \times 3$ matrix written over any “complex” subalgebra of the octonions, then the operation $X \mapsto \overline{M}^T XM$ makes sense, because each entry in $\overline{M}^T XM$ is a sum of terms of the form $m_1x m_2$, and $(m_1x)m_2 = m_1(xm_2)$.
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It is clear that such an operation preserves the determinant if and only if $\det(M) = \pm 1$. 
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It is clear that such an operation preserves the determinant if and only if $\det(M) = \pm 1$.

Suppose now $F = \mathbb{F}_q$. The group $SE_6(q)$ is defined as the group of linear maps which preserve the determinant.
Define the Hermitean form $H$ on $\mathbb{J}_{\mathbb{F}_q^2}$ by

$$H(a, b, c \mid A, B, C) = aa' + bb' + cc' + \text{Tr}(A\overline{A}' + B\overline{B}' + C\overline{C}')$$

where $'$ is an automorphism of $\mathfrak{O}$ induced by the field automorphism $\lambda \mapsto \lambda^q$. 
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where $'$ is an automorphism of $O$ induced by the field automorphism $\lambda \mapsto \lambda^q$.

The group $^2\text{SE}_6(q)$ is defined as the subgroup of $\text{SE}_6(q^2)$ which preserves $H$. 

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Some elements of $\mathbb{SE}_6(q)$

Coordinate permutations, generated by

$$(a, b, c \mid A, B, C) \mapsto (c, a, b \mid C, A, B),$$
$$(a, b, c \mid A, B, C) \mapsto (a, c, b \mid A, C, B)$$

preserve the determinant. These are encoded by

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$
The elements

\[ N_x = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

preserve the determinant.
Some elements of $\text{SE}_6(q)$

The elements

$$N_\mathbf{x} = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

preserve the determinant.

The group contains the diagonal matrices

$$P_u = \begin{pmatrix} u & 0 & 0 \\ 0 & \bar{u} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $u\bar{u} = 1$. 
The action of \( P_u = \text{diag}(u, \bar{u}, 1) \) on the Albert set is given by

\[(a, b, c \mid A, B, C) \mapsto (a, b, c \mid uA, Bu, \bar{u}C\bar{u}).\]

Essentially, the map \( C \mapsto \bar{u}C\bar{u} \) is a product of two reflections and the action generated is that of group \( \text{Spin}_8^+(q) \).
Consider the matrices

\[ Q_{s,t} = \begin{pmatrix} s & t & 0 \\ -t^q & s^q & 0 \\ 0 & 0 & 1 \end{pmatrix}, \]

where \( s, t \in \mathbb{F}_{q^2} \) are such that \( s^{q+1} + t^{q+1} = 1 \). The totality of these matrices is \( q(q^2 - 1) \) and in fact these form a group \( \text{Spin}_3(q) \cong \text{SU}_2(q) \).
Consider the group generated by the matrices $P_u$ and $Q_{s,t}$. Denote by $V_{10}^-$ the subspace $(\lambda, -\lambda^q, 0 \mid 0, 0, C)$ where $\lambda \in \mathbb{F}_{q^2}$ and $C$ is an octonion written over $\mathbb{F}_q$. Let $Q_{10}^-$ be the quadratic form defined on $V_{10}^-$ by

$$Q_{10}^-((\lambda, -\lambda^q, 0 \mid 0, 0, C)) = \lambda \lambda^q + C \bar{C}.$$ 

We notice that $Q_{10}^-$ is of \textit{minus} type.
Consider the group generated by the matrices $P_u$ and $Q_{s,t}$. Denote by $V_{10}$ the subspace $(\lambda, -\lambda^q, 0 \mid 0, 0, C)$ where $\lambda \in \mathbb{F}_{q^2}$ and $C$ is an octonion written over $\mathbb{F}_q$. Let $Q_{10}$ be the quadratic form defined on $V_{10}$ by

$$Q_{10}((\lambda, -\lambda^q, 0 \mid 0, 0, C)) = \lambda \lambda^q + C \overline{C}.$$ 

We notice that $Q_{10}$ is of minus type.

The elements $P_u$ and $Q_{s,t}$ preserve $Q_{10}$ and in fact these generate a group $\text{Spin}_{10}(q)$, except for $q = 2$. 

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Denote $X = (0, 0, 0 | 0, 0, e_1)$ and consider the subgroup $\text{Stab}_G(X)$, where $G = {}^2\text{SE}_6(q)$. In fact, the matrices $P_u$ with the additional constraint $u_{-1} = u_1 = 0$ do the job as well as the matrices $Q_{s,t}$. Such matrices $P_u$ generate $\text{Spin}_6^+(q)$ and combining that with $Q_{s,t}$ we get $\text{Spin}_8^-(q)$. But there is more.

Consider the matrices $M_x$ and $L_x$ defined by

$$
M_x = \begin{pmatrix} 1 & 0 & -\bar{x}' \\ 0 & 1 & 0 \\ x & 0 & 1 \end{pmatrix}, \quad L_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & -\bar{x}' & 1 \end{pmatrix}.
$$

These also stabilise $X$ and generate a subgroup of shape $q^{8+16}$.

Overall we get a subgroup of shape $q^{8+16}.\text{Spin}_8^-(q)$. 
Thank you for your attention!