# On almost recognizability by spectrum of simple classical groups

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Groups $Z_2$ and $Z_2 \times Z_2$ are isospectral.

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- *recognizable by spectrum* (briefly *recognizable*) If $h(G) = 1$;
- *almost recognizable by spectrum* if $h(G) < \infty$;
- *non-recognizable by spectrum* if $h(G) = \infty$.

# Non-recognizability

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If $h(G) = \infty$ then there exists a finite group isospectral to $G$ with nontrivial soluble radical.

So recognizable groups have nonabelian socle.

# Recognition problem

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In 1986 W.J. Shi proved that $h(A_5) = 1$. It was the starting point of the recognition problem for nonabelian simple groups.

# Other examples

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If $q \neq 9$ then $h(PSL_2(q)) = 1$ (Brandl, Shi, 1994).

Let $q = p^k$, where $p$ is a prime, $5 < q \equiv -1 \pmod{6}$,
and $k = 3^r \cdot t$ with $(t, 3) = 1$.
Then $h(PSU_3(q)) = r + 1$ (Zavarnitsine, 2006).

# Restriction on orders

**Theorem (Grechkoseeva, Mazurov, Vasil′ev, 2009)**

If $L$ is a finite simple group, and $G$ is a finite group with $|G| = |L|$ and $\omega(G) = \omega(L)$, then $G \simeq L$.

# Current situation

## Alternating groups

Let $L$ be an alternating group $A_n$, $n \geqslant 5$.

- If $n \notin \{6, 10\}$, then $h(L) = 1$.
- If $n = 6, 10$, then $h(L) = \infty$.

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## Exceptional groups of Lie type

Let $L$ be a simple exceptional group of Lie type, and $\omega(G) = \omega(L)$.

- If $L \neq {}^3D_4(2)$, then $L \leqslant G \leqslant \mathrm{Aut}(L)$ and so $h(L) < \infty$.
- If $L = {}^3D_4(2)$, then $h(L) = \infty$.

# Classical groups

**Theorem (Vasil'ev, Grechkoseeva, 2015)**

*Let $L$ be one of the following nonabelian simple groups:*

- $L_n(q)$, *where* $n \geq 45$ *or* $q$ *is even;*
- $U_n(q)$, *where* $n \geq 45$, *or* $q$ *is even and* $(n, q) \neq (4, 2)$,$(5, 2)$*;*
- $S_{2n}(q)$, $O_{2n+1}(q)$ *where* $n \geq 29$, *or* $q$ *is even,* $n \neq 2, 4$, *and* $(n, q) \neq (3, 2)$*;*
- $O_{2n}^{+}(q)$, *where* $n \geq 31$, *or* $q$ *is even and* $(n, q) \neq (4, 2)$*;*
- $O_{2n}^{-}(q)$, *where* $n \geq 30$, *or* $q$ *is even;*

*Then every finite group isospectral to* $L$ *is isomorphic to some group* $G$ *with* $L \leq G \leq \mathrm{Aut}\, L$. *In particular, there are only finitely many pairwise non-isomorphic finite groups isospectral to* $L$.

# New results

**Theorem (AS, 2016)**

Let $L$ be one of the simple groups $L_n(q)$, $U_n(q)$ with $n \geq 27$, $S_{2n}(q)$, $O_{2n+1}(q)$ with $n \geq 17$, $L = O^+_{2n}(q)$ with $n \geq 19$, and $L = O^-_{2n}(q)$ with $n \geq 18$. Then every finite group isospectral to $L$ is isomorphic to some group $G$ with $L \leq G \leq \mathrm{Aut}\, L$. In particular, there are only finitely many pairwise non-isomorphic finite groups isospectral to $L$.

# Prime graph

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Denote by $t(G)$ the greatest size of a coclique in $GK(G)$.

Denote by $t(2, G)$ the greatest size of a coclique in $GK(G)$ containing 2.

# Main tool

**Theorem (Vasil′ev, 2005)**

Let $G$ be a finite group satisfying the conditions:
(a) there exist three primes in $\pi(G)$ pairwise nonadjacent in $GK(G)$; i.e., $t(G) \geq 3$;
(b) there exists an odd prime in $\pi(G)$ nonadjacent in $GK(G)$ to the prime 2; i.e., $t(2, G) \geq 2$.
Then there is a finite nonabelian simple group $S$ such that $S \leq G/K \leq \mathrm{Aut}(S)$ for the soluble radical $K$ of $G$. Furthermore, $t(S) \geq t(G) - 1$.

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Let $L$ be a simple exceptional group of Lie type. Then $t(L) \leq 12$.

Let $L$ be a simple classical groups from the hypothesis of Theorem. Then $t(L) \geq 14$.

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Suppose that $G$ is a finite group such that $\omega(G) = \omega(L)$, where $L$ is a simple classical group as above. Then there exists a finite nonabelian simple group $S$ such that $S \leq G/K \leq \mathrm{Aut}(S)$ for the soluble radical $K$ of $G$. Moreover, $t(S) \geq t(L) - 1 \geq 13$. Thus $S$ is not exceptional.

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Further, we show that $S \simeq L$ and $K = 1$.

## Conjecture (Grechkoseeva, Vasil′ev, 2015)

*Let* $L$ *be one of the following groups:*

- $L_n(q)$, *where* $n \geq 5$;
- $U_n(q)$, *where* $n \geq 5$ *and* $(n, q) \neq (5, 2)$;
- $S_{2n}(q)$, *where* $n \geq 3$, $n \neq 4$ *and* $(n, q) \neq (3, 2)$;
- $O_{2n+1}(q)$, *where* $q$ *is odd,* $n \geq 3$, $n \neq 4$ *and* $(n, q) \neq (3, 3)$;
- $O^{\varepsilon}_{2n}(q)$, *where* $n \geq 4$ *and* $(n, q, \varepsilon) \neq (4, 2, +), (4, 3, +)$.

*Then every finite group isospectral to* $L$ *is isomorphic to some group* $G$ *with* $L \leq G \leq \mathrm{Aut}\, L$.