Asymptotic density of test elements in free groups and surface groups

Ilir Snopche

Universidade Federal do Rio de Janeiro

Joint work with S. Tanushevski

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- Let $F(x_1, x_2)$ be a free group with basis $\{x_1, x_2\}$ and suppose that $\varphi : F(x_1, x_2) \to F(x_1, x_2)$ is an endomorphism such that $\varphi([x_1, x_2]) = [x_1, x_2]$.

Nielsen's result

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- (Nielsen, 1918) φ must be an automorphism.

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- (Rips, 1981) $[x_1, x_2, ..., x_n]$ is a test element of $F(x_1, ..., x_n)$.
- (Zieschang, 1965) $x_1^k x_2^k \cdots x_n^k$ is a test element of $F(x_1, ..., x_n)$ whenever $k \geq 2$.

- Recall that a retract of a group G is a subgroup $H \leq G$ for which there exists an epimorphism $r : G \to H$ that restricts to the identity homomorphism on H; such epimorphism r is called retraction.

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Theorem (Turner, 1996)

The test elements of a free group F of finite rank are exactly the elements not contained in any proper retract of F.

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- (Groves, 2012) torsion free hyperbolic groups.

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- We denote by $B_X(r) = \{g \in G \mid d_X(e, g) = |g|_X \leq r\}$ the ball of radius $r \geq 0$ centered at the identity in the metric space (G, d_X).

- Given $S \subseteq G$, the asymptotic density of S in G with respect to X is defined as

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- A subset S of G is <u>generic</u> in G (with respect to X) if $\rho_X(S) = 1$, and it is <u>negligible</u> if $\rho_X(S) = 0$.
- Most of the subsets of free groups for which the asymptotic density has been studied and which could be defined by a natural algebraic property are either generic or negligible.

A question

Theorem (Kapovich - Rivin - Schupp - Shpilrain, 2005)

Let $\mathcal{T}$ be the set of test elements of a free group $F(x_1, x_2)$ of rank 2. Then $\mathcal{T}$ has intermediate density (different from 0 and 1). More precisely,

$$\frac{4}{9}(1 - \frac{6}{\pi^2}) \leq \overline{\rho}_{\{x_1,x_2\}}(\mathcal{T}) \leq 1 - \frac{8}{3\pi^2}$$

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Question 1 (K - R - S - S, 2005)

Let F be a free group of rank $n \geq 3$. Is the set of test elements of F negligible?

- Let G be a finitely generated group with a finite generating set X. A subset S of G is called a C-net ($0 \leq C < \infty$) with respect to X if

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- Observation: $S \subseteq G$ is a C-net (with respect to X) if and only if there exist elements $g_1, \ldots, g_m \in B_X(C)$ such that $G = Sg_1 \cup \ldots \cup Sg_m$.

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Lemma

Let G be a finitely generated group, X a finite generating set of G, and $S \subseteq G$. Suppose that $G = Sg_1 \cup \ldots \cup Sg_m$ for some $g_1, \ldots, g_m \in B_X(C)$. Then

$$\overline{\rho}_X(S) \geq \liminf_{k \to \infty} \frac{|S \cap B_X(k)|}{|B_X(k)|} \geq \frac{1}{m|B_X(C)|}.$$

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$$\overline{\rho}_X(\mathcal{T}) \geq \liminf_{k\to\infty} \frac{|\mathcal{T} \cap B_X(k)|}{|B_X(k)|} \geq \frac{1}{(2^{n+1}(2^n-1)+1)|B_X(3n-2)|}$$

for every generating set X of G.

The main result - Free groups

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(C) *If X is a basis of F, then*

$$\overline{\rho}_X(\mathcal{T}) \leq 1 - \frac{4n-4}{(2n-1)^2\zeta(n)}.$$

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- There is a natural homomorphism $\jmath_p : G \to \widehat{G}_p$, which sends $g \in G$ to $(gN) \in \widehat{G}_p$.
- If G is residually finite-p, then $\jmath_p$ is an embedding and we identify $\jmath_p(G)$ with G.

Theorem B (S. - Tanushevski, 2015)

The test elements of a finitely generated pro-p group G are exactly the elements not contained in any proper retract of G (i.e., every finitely generated pro-p group is a Turner group).

Proof of Theorem A - Key ingredients

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Theorem C (S. - Tanushevski, 2015)

Let p be a prime, and let G be a finitely generated residually finite-p Turner group. If $g \in G$ is a test element of $\widehat{G}_p$, then g is a test element of G.

Theorem D (S - Tanushevski, 2015)

Let $G = \langle x_1, \ldots, x_{2n} \mid [x_1, x_2] \ldots [x_{2n-1}, x_{2n}] \rangle$ *be an orientable surface group of genus* $n \geq 2$ *and let* $\mathcal{T}$ *be the set of test elements of* G*. Then:*

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(A) $\mathcal{T}$ *is a* $(161n + 8 \cdot 25^n(n-1)(16n+1) + 33)$*-net with respect to every finite generating set of* G*.*

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$$\frac{1}{|B_X(161n + 8 \cdot 25^n(n-1)(16n+1) + 33)|^2}$$

for every generating set X *of* G*.*

Thank You!