The congruence subgroup property for a family of branch groups

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August 11, 2017

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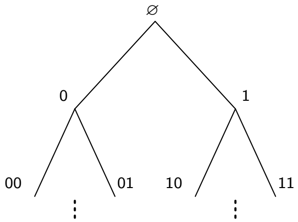

Figure: If $X = \{0, 1\}$ then $\mathcal{T}$ is a binary tree. If $|X| = n$, we say $\mathcal{T}$ is an n-ary tree.

Automorphisms of $\mathcal{T}$

Definition

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Example:

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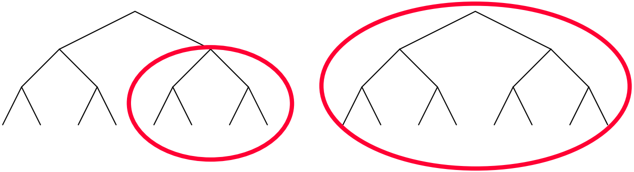

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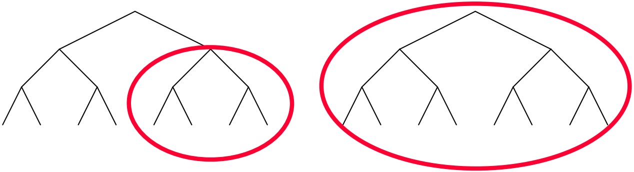

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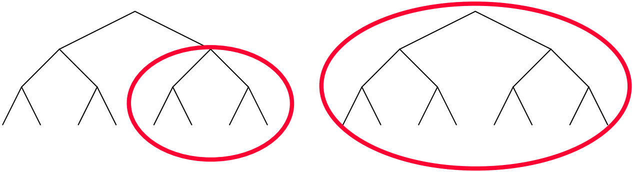

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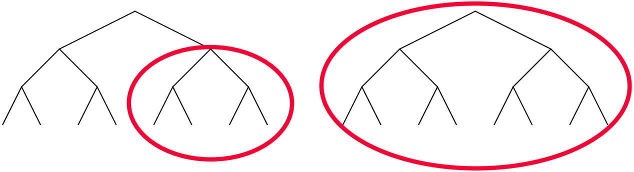
If we have an element g defined by a permutation labeling and similarly zoom in on the $\mathcal{T}_v$, we get another automorphism of $\mathcal{T}$, called *the state of* g *at* v and denoted g_v.

Automorphisms of $\mathcal{T}$



This allows us to decompose g as $(g_1, g_2, \cdots, g_n)g(\varnothing)$ where $n = |X|$, $g(\varnothing)$ is the permutation labeling at $\varnothing$ and each g_i is the state of g at a vertex on the first level.

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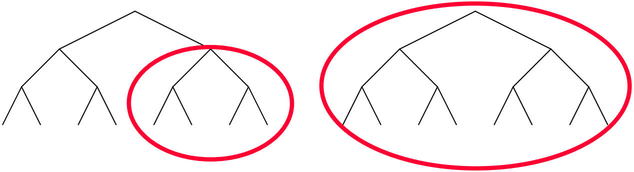

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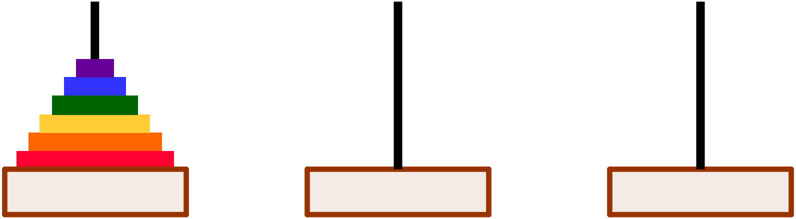
So we get an isomorphism: $Aut(\mathcal{T}) \cong Aut(\mathcal{T}) \wr S_n$ which is $(\prod_n Aut(\mathcal{T})) \rtimes S_n$.

Example

The Hanoi towers group



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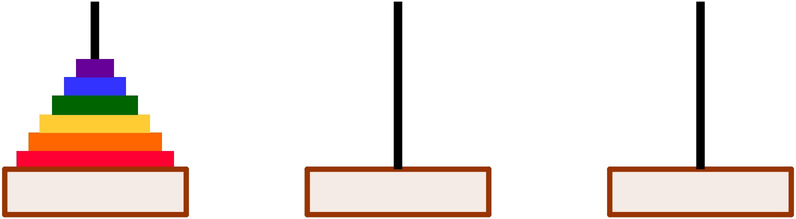

$G_3 = \langle a_1, a_2, a_3 \rangle$

Subgroups of $H \leq \mathcal{T}$

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- For a positive integer m, the mth level stabilizer, $Stab_H(m)$ are the elements of H which stabilizer every vertex on the mth level. An element $g \in Stab_H(m)$ can be described by a tuple $(g_1, \ldots, g_{n^m})_m$ where g_i is the state of g at the ith vertex on the mth level.
- For a positive integer m, the mth level rigid stabilizer, $Rist_H(m)$, is the subgroup of $Stab_H(m)$ consisting of elements of the form $(g_1, \ldots, g_{n^m})_m$ such that for each i, $(1, \ldots, 1, g_i, 1, \ldots, 1)_m$ is also an element of H where g_i is in the ith coordinate.

Example: $(a_1, a_3 a_2, a_1)_1$ is an element in the first level stabilizer of the Hanoi towers group, but not in the rigid stabilizer.

Branch groups

Definition

A group H of automorphisms of $\mathcal{T}$ is a *branch group* if H acts transitively on all levels of $\mathcal{T}$ and for all m, $Rist_H(m)$ has finite index in H.

Branch groups

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Theorem (RS)

$Rist_{G_3}(m) = \prod_{3^m} G_3'$

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Theorem (Bartholdi, Siegenthaler, Zalesskii, 2012)

G_3 does not have property 2.

The groups G_n

For a fixed $n \geq 3$ and for $1 \leq i \leq n$, let σ_i be the permutation $(1, 2, \ldots, i-1, i+1, \ldots, n)$ and let a_i be the automorphism of the n-ary tree defined recursively as $a_i := (1, \ldots, 1, a_i, 1, \ldots, 1)\sigma_i$ where a_i is in the ith coordinate.

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Definition

$G_n := \langle a_1, \ldots, a_n \rangle$.

The groups G_n

Theorem (RS)

For all n, G_n is a branch group. For $n = 4$, $Rist_{G_4}(m) = \prod_{4^m} \langle G_n', a_1 a_3 a_4^2, a_2 a_3, a_1 a_4 \rangle$, a subgroup of index 3. And for all $n \geq 5$ when n is odd $Rist_{G_n}(m) = \prod_{n^m} \{g \mid g \text{ has even word length}\}$, a subgroup of index 2, and when n is even, $Rist_{G_n}(m) = \prod_{n^m} G_n$.

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Theorem (RS)

For $n = 4$ and odd $n \geq 5$, the mth level rigid stabilizer contains the $m+1$ level stabilizer and for even $n \geq 5$ the mth level rigid stabilizer is exactly the mth level stabilizer. Thus G_n has property 2 if and only if $n \neq 3$.

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Corollary (RS)

G_n is just infinite if and only if $n \neq 3$.

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These kernels are invariants of the group (Garrido, 2016)

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Theorem (RS)

For $n \geq 4$ and $d \neq 1$ dividing $(n-1)$, let $H_{n,d}$ be the set of elements of G_n who have a representative with exponent sum on $a_1, \ldots, a_n$ congruent to 0 modulo d. Then $H_{n,d}$ is a subgroup of index d in G_n and is a branch group with non-trivial rigid kernel.

Thank you!

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