Quotients of Coxeter groups associated to signed line graphs.

Robert Shwartz
Ariel University
ISRAEL
Let $G$ be a connected undirected graph without loops, with $n$ vertices and $k$ edges.

Then the line graph of $G$ is the undirected graph $L(G)$, where the following holds:

- Each vertex of $L(G)$ corresponds to a certain edge of $G$;

- Two vertices of $L(G)$ are connected by an edge if the corresponding edges in $G$ have a common endpoint.
For example, if the graph $G$ is:

```
1
\_\_\_\_\_\_\_
2        3        4        5
\_\_\_\_\_\_\_
```

Then the graph $L(G)$ is:

```
13
\_\_\_\_\_\_
12        23        34        46
\_\_\_\_\_\_
```

A vertex $ij$ in $L(G)$ corresponds to the edge of $G$ which connects the vertices $i$ and $j$ of $G$.

Notice that the triangle with vertices 12, 23, 13 in $L(G)$ is induced from the triangle with vertices 1, 2, 3 in $G$,

while all other cycles of $L(G)$ are not induced by a cycle of $G$. 
Signed line graph

Let $f$ be a function from the edges of $L(G)$ to the set $\{-1, +1\}$.

$L(G)_f$ is a signed line graph for the graph $G$, where The edge $e$ of $L(G)_f$ is signed by $f(e)$.

A cycle in a signed graph is called balanced if the product of the values of $f$ along this cycle is equal to $+1$.

A cycle in a signed graph is called non-balanced if the product of the values of $f$ along this cycle is equal to $-1$. 
Signed Coxeter groups

The canonical construction of the standard geometric representation of a simply laced Coxeter group $W$ associated to the Coxeter graph $\Gamma$ can be generalized for a signed Coxeter graph $\Gamma_f$ in the following way.

Let $(W, S)$ be a simply laced Coxeter system, where $S = \{s_1, s_2, ..., s_n\}$, let $\Gamma$ be its Coxeter graph, and let $f$ be a function on edges of $\Gamma$ with values $\pm 1$, i.e., $f(\{s_i, s_j\}) \in \{1, -1\}$ when $m_{ij} = 3$. 
Let us construct the mapping:

- The generator $s_i$ is mapped to the $n \times n$ matrix $\omega_i$ which differs from the identity matrix only by the $i$-th row;

- The $i$-th row of the matrix $\omega_i$ has $-1$ at the position $(i, i)$;

- It has $f\left(\{s_i, s_j\}\right)$ in the position $(i, j)$ when the node $s_j$ is connected to the node $s_i$, i.e., when $m_{ij} = 3$, and it has 0 in the position $(i, j)$ when the nodes $s_j$ and $s_i$ are not connected by an edge, i.e., when $s_j$ and $s_i$ commute.

Thus, we defined the mapping

$$\mathcal{R}_{\Gamma, f} : S \to GL_n(\mathbb{C}), \mathcal{R}_{\Gamma, f}(s_i) = \omega_i.$$
Example:

Consider for example a signed Coxeter graph of the symmetric group $S_4$: $s_1 \xrightarrow{1} s_2 \xrightarrow{-1} s_3$

$$s_1 \mapsto \omega_1 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad s_2 \mapsto \omega_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$s_3 \mapsto \omega_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix}.$$
The following propositions holds:

The matrices $\omega_1, \omega_2, \ldots, \omega_n$ satisfy the Coxeter relations of the group $W$.

The mapping $R_{\Gamma, f}(s_i) = \omega_i$ can be extended to a group homomorphism $W \to GL_n(\mathbb{C})$.

In other words, the matrix group $\Omega_{\Gamma, f} = \langle \omega_1, \omega_2, \ldots, \omega_n \rangle$ is isomorphic to some quotient, may be proper, of the simply laced Coxeter group $W$.

The standard geometric representation is a particular case of the representation $R_{\Gamma, f}$ when the function $f$ maps every edge to 1.
It is natural to inquire how many different (non-isomorphic) matrix groups $\Omega$ can we get this way from a given Coxeter graph $\Gamma$. More precisely:

**Problem.** Given an undirected graph $\Gamma = (V, E)$. To each of $2^{|E|}$ functions from $E$ to $\{1, -1\}$ we associate the matrix group $\Omega_{\Gamma,f}$ as it is described above. How many different groups do we get this way and what can be said about the structure of these groups?
A partial answer to this question was given in the paper:


Let $\Gamma_f = (V, E, f)$ be a signed Coxeter graph. Then the representation $\mathcal{R}_{\Gamma,f}$ is faithful if and only if $\Gamma_f$ is balanced, i.e., if and only if every cycle in the graph has an even number of $-1$'s.

Thus, for all functions $f : E \to \{1, -1\}$ such that the signed graph $\Gamma_f$ is balanced, the group $\Omega_{\Gamma,f}$ is isomorphic to the simply laced Coxeter group associated to the graph $\Gamma$. 
It seems to be a difficult problem to distinguish the cases of non-faithful representations $\mathcal{R}_{\Gamma,f}$.

There is a partial answer to the formulated above problem:

We describe the group $\Omega_{\Gamma,f}$ when $\Gamma$ is a line graph $L(G')$ with certain restriction.
The main Theorem

Let $\Gamma_f$ be a signed graph with $k$ vertices. Assume that $\Gamma = L(G)$, i.e., $\Gamma$ is the line graph of a certain graph $G$ with $n$ vertices and $k$ edges.

Assume that every cycle of $\Gamma_f$, which is not induced from a cycle of $G$, is not balanced.

1. If every cycle of $\Gamma_f$, which is induced from a cycle of $G$, is balanced, then the group $\Omega_{\Gamma,f}$ is isomorphic to a certain semidirect product of $\mathbb{Z}^{(n-1)(k-n+1)}$ with the symmetric group $S_n$.

2. If there exists at least one non-balanced cycle in $\Gamma_f$, which is induced from a cycle of $G$, then the group $\Omega_{\Gamma,f}$ is isomorphic to a certain semidirect product of $\mathbb{Z}^{n(k-n)}$ with the Coxeter group $D_n$. 
In order to prove the Theorem we construct a certain matrix $\alpha$ such that

$$\alpha \cdot \Omega_{\Gamma,f} \cdot \alpha^{-1} = X_{n,k}$$

where the group $X_{n,k} \simeq \mathbb{Z}^{(n-1)(k-n+1)} \rtimes S_n$,

or

$$\alpha \cdot \Omega_{\Gamma,f} \cdot \alpha^{-1} = Y_{n,k}$$

where the group $Y_{n,k} \simeq \mathbb{Z}^{n(k-n)} \rtimes D_n$. 
The subgroup $\mathcal{G}_n$ of $GL_{n-1}(\mathbb{C})$.

A matrix of $\mathcal{G}_n$ is either a certain $(n-1) \times (n-1)$ permutation matrix, or is a matrix which has the following structure:

• For a certain $i \in \{1, 2, ..., n - 1\}$, all the elements of the $i$-th row equal to $-1$;

• There exists $j \in \{1, 2, ..., n-1\}$ such that all the elements of the $j$-th column are zeros except the element in the position $ij$ which is $-1$;

• If we delete the $i$-th row and the $j$-th column we obtain a certain $(n - 2) \times (n - 2)$ permutation matrix.

Then $\mathcal{G}_n$ is a subgroup of $GL_{n-1}(\mathbb{C})$, and $\mathcal{G}_n$ is isomorphic to the symmetric group $S_n$. 

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Example:

The matrices

\[
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-1 & -1 & -1
\end{pmatrix}
\]

generate the subgroup \( \mathcal{G}_4 \) of \( GL_3(\mathbb{C}) \) which is isomorphic to \( S_4 \).
The subgroup $\mathfrak{D}_n$ of $GL_{n-1}(\mathbb{C})$.

Let $\mathfrak{D}_n$ be a subset of $GL_n(\mathbb{C})$, which consists of all matrices having the following structure:

- A matrix of $\mathfrak{D}_n$ has the unique non-zero entry in each row and each column, which is 1 or $-1$;

- The number of $-1$’s is even.

Then $\mathfrak{D}_n$ is a subgroup of $GL_n(\mathbb{C})$, and $\mathfrak{D}_n$ is isomorphic to the Coxeter group $D_n$. 
The subgroup $X_{n,k}$ of $GL_k(\mathbb{C})$.

Let $k$ and $n$ be natural numbers such that $k \geq n - 1$. Let $X_{n,k}$ be the following subset of $GL_k(\mathbb{C})$:

$$X_{n,k} = \left\{ \begin{pmatrix} P & 0_{(n-1) \times (k-n+1)} \\ Q & I_{k-n+1} \end{pmatrix} \right\}$$

such that:

$$P \in \mathfrak{S}_n, \ Q \in \mathbb{Z}^{(k-n+1) \times (n-1)}$$

Then:

- $X_{n,k}$ is a subgroup of $GL_k(\mathbb{C})$;

- $X_{n,k}$ is isomorphic to a semidirect product of $\mathbb{Z}^{(n-1)(k-n+1)}$ with the symmetric group $S_n$. 
The subgroup $Y_{n,k}$ of $GL_k(\mathbb{C})$.

Let $k$ and $n$ be natural numbers such that $k \geq n$. Let $Y_{n,k}$ be the following subset of $GL_k(\mathbb{C})$:

$$Y_{n,k} = \left\{ \begin{pmatrix} P & 0_{n \times (k-n)} \\ Q & I_{k-n} \end{pmatrix} \right\}$$

such that:

$$P \in D_n, \quad Q \in \mathbb{Z}^{(k-n) \times n}$$

Then:

- $Y_{n,k}$ is a subgroup of $GL_k(\mathbb{C})$;

- $Y_{n,k}$ is isomorphic to a semidirect product of $\mathbb{Z}^{n(k-n)}$ with the Coxeter group $D_n$. 
The structure of the conjugating matrix $\alpha$

$\alpha = \mathcal{A}(\Gamma_f) \cdot \mathcal{D}(\Gamma_f)$, where

The matrices $\mathcal{A}(\Gamma_f)$ and $\mathcal{D}(\Gamma_f)$ depends on the graph $\Gamma_f$,

which is the signed line graph of $G$.

Now, we describe the structures of these matrices
Let \( T(G) \) be a spanning tree of the graph \( G \).

Let \( C_1, C_2, \ldots, C_{k-n+1} \) be a certain basis of the binary cycle space of \( G \).

Let \( C'_i(\Gamma) \) be the cycle of \( \Gamma_f \) which is induced from the cycle \( C_i(G) \).

The vertices of \( C'_i(\Gamma) \) correspond to the edges of \( C_i(G) \) in \( G \).

Consider two cases:

- **Case 1** - Every cycle \( C'_i(\Gamma) \) is a balanced cycle in \( \Gamma_f \);

- **Case 2** - There exists at least one non-balanced cycle \( C'_i(\Gamma) \) in \( \Gamma_f \). In this case, without loss of generality, assume that \( C'_1(\Gamma) \) is a non-balanced cycle in \( \Gamma_f \).
Case 1:

For $1 \leq i \leq n$, denote by $v_i$ the vertices of $G$.

Assign the numbers $1, 2, \ldots, n-1$ to the $n-1$ edges of $T(G)$ in such a way that a vertex $v_i$ is an endpoint of the edge $e_i$.

Notice that such an indexing of edges of $T(G)$ is unique for a fixed indexing of vertices of $G$.

Let us index the remained $k-n+1$ edges of $G$ in the following way:

- $e_n$ should belong to the cycle $C_1(G')$,
- $e_{n+1}$ should belong to $C_2(G)$, \ldots, 
- $e_k$ should belong to $C_{k-n+1}(G)$.

Let $\ell_i$ be the vertex of $\Gamma_f$ which corresponds to the edge $e_i$ of $G$. 
The matrix $\mathcal{A}(\Gamma_f)$

Let $\mathcal{A}(\Gamma_f)$ be a $k \times k$ matrix defined as follows:

- $\mathcal{A}(\Gamma_f)_{i,i} = 1$ for every $1 \leq i \leq k$;

- $\mathcal{A}(\Gamma_f)_{i,j} = -f(\ell_i, \ell_j)$ when the edges $e_i$ and $e_j$ in $G$ have a common endpoint $v_i$, and $1 \leq i \leq n - 1$;

- $\mathcal{A}(\Gamma_f)_{i,j} = 0$ otherwise.

The matrix $\mathcal{A}(\Gamma_f)$ is an invertible matrix with determinant 1.
The matrix $\mathcal{D}(\Gamma_f)$

Let $\mathcal{D}(\Gamma_f)$ be a $k \times k$ diagonal matrix defined as follows:

1. For $n \leq i \leq k$, $\mathcal{D}(\Gamma_f)_{i,i} = 1$;

2. For $1 \leq i \leq n-1$, $\mathcal{D}(\Gamma_f)_{i,i} = (-1)^{\partial_i}$, where $(\partial_1, \partial_2, \ldots, \partial_{n-1})$ is a solution for the following system of the linear equations over $\mathbb{F}_2$: 

$$
\begin{align*}
\sum_{j=1}^{n-1} \partial_j &= 0 \\
\sum_{j=1}^{n-1} \partial_j^2 &= 0 \\
\sum_{j=1}^{n-1} \partial_j^3 &= 0 \\
&\vdots \\
\sum_{j=1}^{n-1} \partial_j^{n-1} &= 0
\end{align*}
$$
• $d_i + d_j = \tilde{f}(\{\ell_i, \ell_j\})$ when the endpoints of $e_j$ in $G$ are $v_i$ and $v_j$;

• $d_i + d_j = 1 + \tilde{f}(\{\ell_i, \ell_j\})$ when $v_n$ is the common endpoint of $e_i$ and $e_j$ in $G$.

where:

\[
\tilde{f}(\{\ell_i, \ell_j\}) = \begin{cases} 
1 , & f(\{\ell_i, \ell_j\}) = -1 \\
0 , & f(\{\ell_i, \ell_j\}) = 1 
\end{cases}
\]
Case 2:

For $1 \leq i \leq n$, denote by $v_i$ the vertices of $G$ (like in case 1).

Assign the numbers 1, 2, ..., $n$ to the $n$ edges of $T(G) \cup C_1(G)$ in such a way that a vertex $v_i$ is an endpoint of the edge $e_i$.

Let us index the remained $k - n$ edges of $G$ in the following way:

- $e_{n+1}$ should belong to the cycle $C_2(G)$,
- $e_{n+2}$ should belong to $C_3(G)$, ...
- $e_k$ should belong to $C_{k-n+1}(G)$.

Similarly to Case 1, let $\ell_i$ be the vertex of $\Gamma_f$ which corresponds to the edge $e_i$ of $G$. 

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Let $A(\Gamma_f)$ be a $k \times k$ matrix defined as follows:

- $A(\Gamma_f)_{i,i} = 1$ for every $1 \leq i \leq k$;

- $A(\Gamma_f)_{i,j} = -f(\ell_i, \ell_j)$ when the edges $e_i$ and $e_j$ in $G$ have a common endpoint $v_i$, and $1 \leq i \leq n$;

- $A(\Gamma_f)_{i,j} = 0$ otherwise.

In Case 2, $\alpha = A(\Gamma_f)$. 
Example 1:

Let $G$ be the following graph:

```
   2
  /\  \\
(3)-e_2-e_4-(1)
  \  /  \\
  \ e_3 /  \\
   4-\-e_5
```

Then, $T(G)$ as follows:

```
   2
  /\  \\
(3)-e_2-e_1-(1)
  \  /  \\
  \ e_3 /  \\
   4-\-e_5
```
Γ_f, where

- \( f(\{\ell_1, \ell_2\}) = f(\{\ell_2, \ell_4\}) = f(\{\ell_1, \ell_4\}) = f(\{\ell_3, \ell_4\}) = f(\{\ell_3, \ell_5\}) = f(\{\ell_4, \ell_5\}) = (+1) \);

- \( f(\{\ell_2, \ell_3\}) = f(\{\ell_1, \ell_5\}) = (-1) \).
Thus:

• $\tilde{f} (\{\ell_1, \ell_2\}) = 0$, since $f (\{\ell_1, \ell_2\}) = 1$;

• $\tilde{f} (\{\ell_2, \ell_3\}) = 1$, since $f (\{\ell_2, \ell_3\}) = -1$.

Therefore the following equations holds in $\mathbb{F}_2$:

• $d_1 + d_2 = \tilde{f} (\{\ell_1, \ell_2\}) = 0$;

• $d_2 + d_3 = \tilde{f} (\{\ell_2, \ell_3\}) = 1$

where the solutions are:

• $(d_1, d_2, d_3) = (-1, -1, 1)$;

• $(d_1, d_2, d_3) = (1, 1, -1)$.
Thus:

\[
\mathcal{D}(\Gamma_f) = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

or

\[
\mathcal{D}(\Gamma_f) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
\mathcal{A}(\Gamma_f) = \begin{pmatrix}
1 & 0 & 0 & -1 & 1 \\
-1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]
\( \Omega_{\Gamma_f} \) is generated by:

\[
\begin{align*}
\omega_1 &= \begin{pmatrix}
-1 & 1 & 0 & 1 & -1 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix} \\
\omega_2 &= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & -1 & -1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix} \\
\omega_3 &= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & -1 & -1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{align*}
\]
\[
\omega_4 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 
\end{pmatrix}
\]

\[
\omega_5 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
-1 & 0 & 1 & 1 & -1 
\end{pmatrix}
\]
\( (A(\Gamma_f) \cdot \omega_1 \cdot (A(\Gamma_f)^{-1} = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix} \)

\( (A(\Gamma_f) \cdot \omega_2 \cdot (A(\Gamma_f)^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix} \)

\( (A(\Gamma_f) \cdot \omega_3 \cdot (A(\Gamma_f)^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix} \)
\[(A(\Gamma_f) \cdot \omega_4 \cdot (A(\Gamma_f)^{-1} = \begin{pmatrix}
0 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 
\end{pmatrix}
\]

\[ (A(\Gamma_f) \cdot \omega_5 \cdot (A(\Gamma_f)^{-1} = \begin{pmatrix}
-1 & -1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
-2 & -1 & 1 & 0 & 1 
\end{pmatrix}
\]
Now, conjugating by \( \mathcal{D}(\Gamma_f) \):

\[
(\mathcal{D}(\Gamma_f) \cdot (A(\Gamma_f) \cdot \omega_1 \cdot (A(\Gamma_f))^{-1} \cdot (\mathcal{D}(\Gamma_f))^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\]

\[
(\mathcal{D}(\Gamma_f) \cdot (A(\Gamma_f) \cdot \omega_2 \cdot (A(\Gamma_f))^{-1} \cdot (\mathcal{D}(\Gamma_f))^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\]

\[
(\mathcal{D}(\Gamma_f) \cdot (A(\Gamma_f) \cdot \omega_3 \cdot (A(\Gamma_f))^{-1} \cdot (\mathcal{D}(\Gamma_f))^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\]

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\[
(D(\Gamma_f) \cdot (A(\Gamma_f) \cdot \omega_4 \cdot (A(\Gamma_f)^{-1} \cdot (D(\Gamma_f)^{-1} = \\
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

or

\[
(D(\Gamma_f) \cdot (A(\Gamma_f) \cdot \omega_4 \cdot (A(\Gamma_f)^{-1} \cdot (D(\Gamma_f)^{-1} = \\
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]
\((\mathcal{D}(\Gamma_f) \cdot (\mathcal{A}(\Gamma_f) \cdot \omega_5 \cdot (\mathcal{A}(\Gamma_f)^{-1} \cdot (\mathcal{D}(\Gamma_f)^{-1} = \begin{pmatrix} -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 1 \end{pmatrix}\)

or

\((\mathcal{D}(\Gamma_f) \cdot (\mathcal{A}(\Gamma_f) \cdot \omega_5 \cdot (\mathcal{A}(\Gamma_f)^{-1} \cdot (\mathcal{D}(\Gamma_f)^{-1} = \begin{pmatrix} -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -2 & -1 & -1 & 0 & 1 \end{pmatrix}\)
The two different choices of \((\mathcal{D}(\Gamma_f))\), gives different matrices for
\[
(\mathcal{D}(\Gamma_f) \cdot (\mathcal{A}(\Gamma_f) \cdot \omega_4 \cdot (\mathcal{A}(\Gamma_f)^{-1} \cdot (\mathcal{D}(\Gamma_f)^{-1}
\]
and for
\[
(\mathcal{D}(\Gamma_f) \cdot (\mathcal{A}(\Gamma_f) \cdot \omega_5 \cdot (\mathcal{A}(\Gamma_f)^{-1} \cdot (\mathcal{D}(\Gamma_f)^{-1},
\]
where the difference is in the last two rows of the matrices, such:

If in the first choice of \((\mathcal{D}(\Gamma_f))\), the \(2 \times 3\) downer left sub-matrix is \(Q\), then

in the second choice of \((\mathcal{D}(\Gamma_f))\), the \(2 \times 3\) downer left sub-matrix is \(-Q\).
Example 2:

Let $G$ be the following graph:

```
  2                     1
 / \                    / \        e1  e2  e3
 3---\---1               \--\---  e4  e5
    \--\                     \--
     4                       1
```

Then $C_1$ is:

```
  2                     1
 / \                    / \        e1  e2  e3
 3---\---1               \--\---  e4  e5
    \--\                     \--
     4                       1
```

and $T(G)$ is:

```
  2                     1
 / \                    / \        e1  e2  e3
 3---\---1               \--\---  e4  e5
    \--\                     \--
     4                       1
```
Then, $T(G) \cap C_1$ as follows:

```
    2
   /\  \
  /   \
3----1
   \   /
    \ / \\
     4
```

$e_2$, $e_1$, $e_3$, $e_4$
\( \Gamma_f \), where

- \( f (\{\ell_1, \ell_2\}) = f (\{\ell_2, \ell_3\}) = f (\{\ell_3, \ell_4\}) = f (\{\ell_4, \ell_5\}) = f (\{\ell_3, \ell_5\}) = f (\{\ell_1, \ell_5\}) = (+1) \);

- \( f (\{\ell_2, \ell_4\}) = f (\{\ell_1, \ell_3\}) = (-1) \).
Therefore:

\[ A(\Gamma_f) = \begin{pmatrix} 1 & 0 & 1 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \]
\( \Omega_{\Gamma_f} \) is generated by:

\[
\begin{align*}
\omega_1 &= \begin{pmatrix}
-1 & 1 & -1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix} \\
\omega_2 &= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & -1 & 1 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix} \\
\omega_3 &= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
-1 & 1 & -1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]
\[ \omega_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \]

\[ \omega_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & -1 \end{pmatrix} \]
\[
\begin{align*}
A(\Gamma_f) \cdot \omega_1 \cdot A(\Gamma_f)^{-1} &= \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix} \\
A(\Gamma_f) \cdot \omega_2 \cdot A(\Gamma_f)^{-1} &= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix} \\
A(\Gamma_f) \cdot \omega_3 \cdot A(\Gamma_f)^{-1} &= \begin{pmatrix}
0 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\end{align*}
\]
\[ A(\Gamma_f) \cdot \omega_4 \cdot A(\Gamma_f)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \]

\[ A(\Gamma_f) \cdot \omega_5 \cdot A(\Gamma_f)^{-1} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix} \]