Quotients of Coxeter groups associated to signed line graphs.

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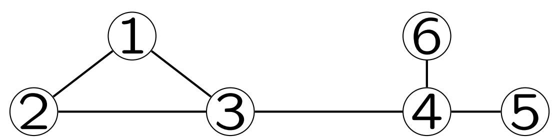
Let G be a connected undirected graph without loops, with n vertices and k edges.

Then the line graph of G is the undirected graph $L(G)$, where the following holds:

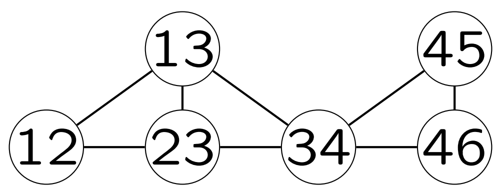
- Each vertex of $L(G)$ corresponds to a certain edge of G;

- Two vertices of $L(G)$ are connected by an edge if the corresponding edges in in G have a common endpoint.

For example, if the graph G is:



Then the graph $L(G)$ is:



A vertex ij in $L(G)$ corresponds to the edge of G which connects the vertices i and j of G.

Notice that the triangle with vertices 12, 23, 13 in $L(G)$ is induced from the triangle with vertices 1, 2, 3 in G,

while all other cycles of $L(G)$ are not induced by a cycle of G.

Signed line graph

Let f be a function from the edges of $L(G)$ to the set $\{-1, +1\}$.

$L(G)_f$ is a signed line graph for the graph G, where The edge e of $L(G)_f$ is signed by $f(e)$.

A cycle in a signed graph is called *balanced* if the product of the values of f along this cycle is equal to $+1$.

A cycle in a signed graph is called *non-balanced* if the product of the values of f along this cycle is equal to -1.

Signed Coxeter groups

The canonical construction of the standard geometric representation of a simply laced Coxeter group W assoiated to the Coxeter graph Γ can be generalized for a signed Coxeter graph Γ_f in the following way.

Let (W, S) be a simply laced Coxeter system, where $S = \{s_1, s_2, ..., s_n\}$, let Γ be its Coxeter graph, and let f be a function on edges of Γ with values ± 1, i.e., $f\left(\left\{s_i, s_j\right\}\right) \in \{1, -1\}$ when $m_{ij} = 3$.

Let us construct the mapping:

- The generator s_i is mapped to the $n \times n$ matrix ω_i which differs from the identity matrix only by the i-th row;

- The i-th row of the matrix ω_i has -1 at the position (i, i);

- It has $f\left(\left\{s_i, s_j\right\}\right)$ in the position (i, j) when the node s_j is connected to the node s_i, i.e., when $m_{ij} = 3$, and it has 0 in the position (i, j) when the nodes s_j and s_i are not connected by an edge, i.e., when s_j and s_i commute.

Thus, we defined the mapping
$\mathcal{R}_{\Gamma,f} : S \to GL_n(\mathbb{C})$, $\mathcal{R}_{\Gamma,f}(s_i) = \omega_i$.

Example:

Consider for example a signed Coxeter graph of the symmetric group S_4: $s_1 \overset{1}{\text{———}} s_2 \overset{-1}{\text{———}} s_3$

$$s_1 \mapsto \omega_1 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; s_2 \mapsto \omega_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$s_3 \mapsto \omega_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix}.$$

The following propositions holds:

The matrices ω_1, ω_2,..., ω_n satisfy the Coxeter relations of the group W.

The mapping $\mathcal{R}_{\Gamma,f}(s_i) = \omega_i$ can be extended to a group homomorphism $W \to GL_n(\mathbb{C})$.

In other words, the matrix group $\Omega_{\Gamma,f} = \langle \omega_1, \omega_2, ..., \omega_n \rangle$ is isomorphic to some quotient, may be proper, of the simply laced Coxeter group W.

The standard geometric representation is a particular case of the representation $\mathcal{R}_{\Gamma,f}$ when the function f maps every edge to 1.

It is natural to inquire how many different (non-isomorphic) matrix groups Ω can we get this way from a given Coxeter graph Γ. More precisely:

Problem. Given an undirected graph $\Gamma = (V, E)$. To each of $2^{|E|}$ functions from E to $\{1, -1\}$ we associate the matrix group $\Omega_{\Gamma,f}$ as it is described above. How many different groups do we get this way and what can be said about the structure of these groups?

A partial answer to this question was given in the paper:

V. Bugaenko, Y. Cherniavsky, T. Nagnibeda, R. Shwartz,"Weighted Coxeter graphs and generalized geometric representations of Coxeter groups", Discrete Applied Mathematics 192 (2015)

Let $\Gamma_f = (V, E, f)$ be a signed Coxeter graph. Then the representation $\mathcal{R}_{\Gamma,f}$ is faithful if and only if Γ_f is balanced,
i.e., if and only if every cycle in the graph has an even number of -1's.

Thus, for all functions $f : E \to \{1, -1\}$ such that the signed graph Γ_f is balanced, the group $\Omega_{\Gamma,f}$ is isomorphic to the simply laced Coxeter group associated to the graph Γ.

It seems to be a difficult problem to distinguish the cases of non-faithful representations $\mathcal{R}_{\Gamma,f}$.

There is a partial answer to the formulated above problem:

We describe the group $\Omega_{\Gamma,f}$ when Γ is a line graph $L(G)$ with certain restriction.

The main Theorem

Let Γ_f be a signed graph with k vertices. Assume that $\Gamma = L(G)$, i.e., Γ is the line graph of a certain graph G with n vertices and k edges.

Assume that every cycle of Γ_f, which is not induced from a cycle of G, is not balanced.

1. If every cycle of Γ_f, which is induced from a cycle of G, is balanced, then the group $\Omega_{\Gamma,f}$ is isomorphic to a certain semidirect product of $\mathbb{Z}^{(n-1)(k-n+1)}$ with the symmetric group S_n.

2. If there exists at least one non-balanced cycle in Γ_f, which is induced from a cycle of G, then the group $\Omega_{\Gamma,f}$ is isomorphic to a certain semidirect product of $\mathbb{Z}^{n(k-n)}$ with the Coxeter group D_n.

In order to prove the Theorem we construct a certain matrix α such that

$$\alpha \cdot \Omega_{\Gamma,f} \cdot \alpha^{-1} = X_{n,k}$$

where the group $X_{n,k} \simeq \mathbb{Z}^{(n-1)(k-n+1)} \rtimes S_n$,

or

$$\alpha \cdot \Omega_{\Gamma,f} \cdot \alpha^{-1} = Y_{n,k}$$

where the group $Y_{n,k} \simeq \mathbb{Z}^{n(k-n)} \rtimes D_n$.

The subgroup $\mathfrak{S}_n$ of $GL_{n-1}(\mathbb{C})$.

A matrix of $\mathfrak{S}_n$ is either a certain $(n-1)\times(n-1)$ permutation matrix,

or is a matrix which has the following structure:

- For a certain $i \in \{1, 2, ..., n-1\}$, all the elements of the i-th row equal to -1;

- There exists $j \in \{1, 2, ..., n-1\}$ such that all the elements of the j-th column are zeros except the element in the position ij which is -1;

- If we delete the i-th row and the j-th column we obtain a certain $(n-2) \times (n-2)$ permutation matrix.

Then $\mathfrak{S}_n$ is a subgroup of $GL_{n-1}(\mathbb{C})$, and $\mathfrak{S}_n$ is isomorphic to the symmetric group S_n.

Example:

The matrices

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & -1 \end{pmatrix}$$

generate the subgroup $\mathfrak{S}_4$ of $GL_3(\mathbb{C})$ which is isomorphic to S_4.

The subgroup $\mathfrak{D}_n$ of $GL_{n-1}(\mathbb{C})$.

Let $\mathfrak{D}_n$ be a subset of $GL_n(\mathbb{C})$, which consists of all matrices having the following structure:

- A matrix of $\mathfrak{D}_n$ has the unique non-zero entry in each row and each column, which is 1 or -1;
- The number of -1's is even.

Then $\mathfrak{D}_n$ is a subgroup of $GL_n(\mathbb{C})$, and

$\mathfrak{D}_n$ is isomorphic to the Coxeter group D_n.

The subgroup $X_{n,k}$ of $GL_k(\mathbb{C})$.

Let k and n be natural numbers such that $k \geqslant n-1$. Let $X_{n,k}$ be the following subset of $GL_k(\mathbb{C})$:

$$X_{n,k} = \left\{ \begin{pmatrix} P & 0_{(n-1)\times(k-n+1)} \\ Q & I_{k-n+1} \end{pmatrix} \right\}$$

such that:

$$P \in \mathfrak{S}_n \, , \, Q \in \mathbb{Z}^{(k-n+1)\times(n-1)}$$

Then:

- $X_{n,k}$ is a subgroup of $GL_k(\mathbb{C})$;

- $X_{n,k}$ is isomorphic to a semidirect product of $\mathbb{Z}^{(n-1)(k-n+1)}$ with the symmetric group S_n.

The subgroup $Y_{n,k}$ of $GL_k(\mathbb{C})$.

Let k and n be natural numbers such that $k \geqslant n$. Let $Y_{n,k}$ be the following subset of $GL_k(\mathbb{C})$:

$$Y_{n,k} = \left\{ \begin{pmatrix} P & 0_{n\times(k-n)} \\ Q & I_{k-n} \end{pmatrix} \right\}$$

such that:

$$P \in \mathfrak{D}_n \, , \, Q \in \mathbb{Z}^{(k-n)\times n}$$

Then:

- $Y_{n,k}$ is a subgroup of $GL_k(\mathbb{C})$;

- $Y_{n,k}$ is isomorphic to a semidirect product of $\mathbb{Z}^{n(k-n)}$ with the Coxeter group D_n.

The structure of the conjugating matrix α

$\alpha = \mathcal{A}(\Gamma_f) \cdot \mathcal{D}(\Gamma_f)$, where

The matrices $\mathcal{A}(\Gamma_f)$ and $\mathcal{D}(\Gamma_f)$ depends on the graph Γ_f,

which is the signed line graph of G.

Now, we describe the structures of these matrices

Let $T(G)$ be a spanning tree of the graph G.

Let C_1, C_2, ... , C_{k-n+1} be a certain basis of the binary cycle space of G.

Let $C'_i(\Gamma)$ be the cycle of Γ_f which is induced from the cycle $C_i(G)$.

The vertices of $C'_i(\Gamma)$ correspond to the edges of $C_i(G)$ in G.

Consider two cases:

- **Case 1** - Every cycle $C'_i(\Gamma)$ is a balanced cycle in Γ_f;

- **Case 2** - There exists at least one non-balanced cycle $C'_i(\Gamma)$ in Γ_f. In this case, without loss of generality, assume that $C'_1(\Gamma)$ is a non-balanced cycle in Γ_f.

Case 1:.

For $1 \leq i \leq n$, denote by v_i the vertices of G.

Assign the numbers 1, 2, ... , $n-1$ to the $n-1$ edges of $T(G)$
in such a way that a vertex v_i is an endpoint of the edge e_i.

Notice that such an indexing of edges of $T(G)$ is unique for a fixed indexing of vertices of G.

Let us index the remained $k - n + 1$ edges of G in the following way:

e_n should belong to the cycle $C_1(G)$,
e_{n+1} should belong to $C_2(G)$, ... ,
e_k should belong to $C_{k-n+1}(G)$.

Let ℓ_i be the vertex of Γ_f which corresponds to the edge e_i of G.

The matrix $\mathcal{A}(\Gamma_f)$

Let $\mathcal{A}(\Gamma_f)$ be a $k \times k$ matrix defined as follows:

- $\mathcal{A}(\Gamma_f)_{i,i} = 1$ for every $1 \leq i \leq k$;

- $\mathcal{A}(\Gamma_f)_{i,j} = -f\left(\ell_i, \ell_j\right)$ when the edges e_i and e_j in G have a common endpoint v_i, and $1 \leq i \leq n-1$;

- $\mathcal{A}(\Gamma_f)_{i,j} = 0$ otherwise.

The matrix $\mathcal{A}(\Gamma_f)$ is an invertible matrix with determinant 1.

The matrix $\mathcal{D}(\Gamma_f)$

Let $\mathcal{D}(\Gamma_f)$ be a $k \times k$ diagonal matrix defined as follows:

- For $n \leq i \leq k$, $\mathcal{D}(\Gamma_f)_{i,i} = 1$;
- For $1 \leq i \leq n-1$, $\mathcal{D}(\Gamma_f)_{i,i} = (-1)^{\mathfrak{d}_i}$, where $(\mathfrak{d}_1, \mathfrak{d}_2, \ldots, \mathfrak{d}_{n-1})$ is a solution for the following system of the linear equations over $\mathbb{F}_2$:

- $d_i + d_j = \tilde{f}\left(\left\{\ell_i, \ell_j\right\}\right)$ when the endpoints of e_j in G are v_i and v_j;

- $d_i + d_j = 1 + \tilde{f}\left(\left\{\ell_i, \ell_j\right\}\right)$ when v_n is the common endpoint of e_i and e_j in G.

where:

$$\tilde{f}\left(\left\{\ell_i, \ell_j\right\}\right) = \begin{cases} 1\ , & f\left(\left\{\ell_i, \ell_j\right\}\right) = -1 \\ 0\ , & f\left(\left\{\ell_i, \ell_j\right\}\right) = 1 \end{cases}$$

Case 2:

For $1 \leq i \leq n$, denote by v_i the vertices of G (like in case 1).

Assign the numbers 1, 2, ... , n to the n edges of $T(G) \cup C_1(G)$
in such a way that a vertex v_i is an endpoint of the edge e_i.

Let us index the remained $k - n$ edges of G in the following way:

e_{n+1} should belong to the cycle $C_2(G)$,
e_{n+2} should belong to $C_3(G)$, ... ,
e_k should belong to $C_{k-n+1}(G)$.

Similarly to Case 1, let ℓ_i be the vertex of Γ_f which corresponds to the edge e_i of G.

Let $\mathcal{A}(\Gamma_f)$ be a $k \times k$ matrix defined as follows:

- $\mathcal{A}(\Gamma_f)_{i,i} = 1$ for every $1 \leq i \leq k$;

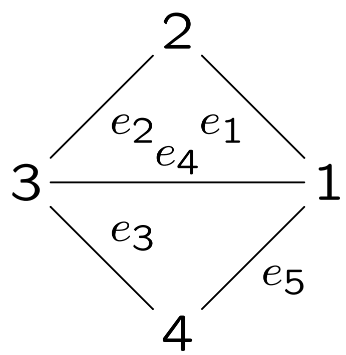
- $\mathcal{A}(\Gamma_f)_{i,j} = -f\left(\ell_i, \ell_j\right)$ when the edges e_i and e_j in G have a common endpoint v_i, and $1 \leq i \leq n$;

- $\mathcal{A}(\Gamma_f)_{i,j} = 0$ otherwise.

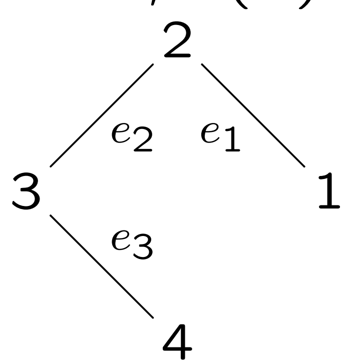
In Case 2, $\alpha = \mathcal{A}(\Gamma_f)$.

Example 1:

Let G be the following graph:



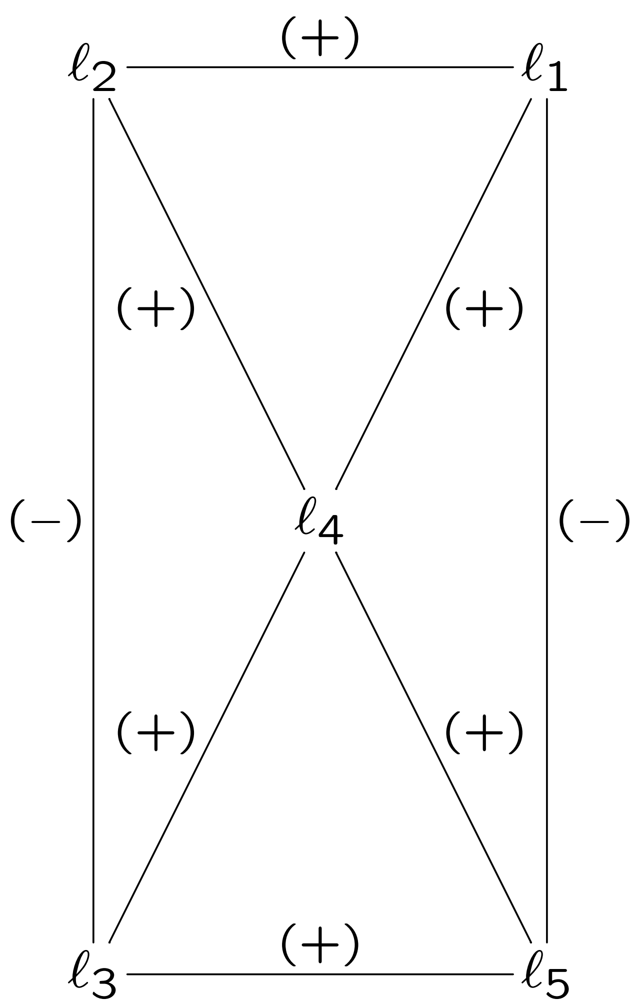
Then, $T(G)$ as follows:



Γ_f, where

- $f(\{\ell_1,\ell_2\}) = f(\{\ell_2,\ell_4\}) = f(\{\ell_1,\ell_4\}) = f(\{\ell_3,\ell_4\}) = f(\{\ell_3,\ell_5\}) = f(\{\ell_4,\ell_5\}) = (+1)$;

- $f(\{\ell_2,\ell_3\}) = f(\{\ell_1,\ell_5\}) = (-1)$.



Thus:

- $\tilde{f}(\{\ell_1, \ell_2\}) = 0$, since $f(\{\ell_1, \ell_2\}) = 1$;

- $\tilde{f}(\{\ell_2, \ell_3\}) = 1$, since $f(\{\ell_2, \ell_3\}) = -1$.

Therefore the following equations holds in $\mathbb{F}_2$:

- $d_1 + d_2 = \tilde{f}(\{\ell_1, \ell_2\}) = 0$;

- $d_2 + d_3 = \tilde{f}(\{\ell_2, \ell_3\}) = 1$

where the solutions are:

- $(\mathfrak{d}_1, \mathfrak{d}_2, \mathfrak{d}_3) = (-1, -1, 1)$;

- $(\mathfrak{d}_1, \mathfrak{d}_2, \mathfrak{d}_3) = (1, 1, -1)$.

Thus:

$$\mathcal{D}(\Gamma_f) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

or

$$\mathcal{D}(\Gamma_f) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{A}(\Gamma_f) = \begin{pmatrix} 1 & 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Ω_{Γ_f} is generated by:

$$\omega_1 = \begin{pmatrix} -1 & 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\omega_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\omega_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\omega_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\omega_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 1 & -1 \end{pmatrix}$$

$$(\mathcal{A}(\Gamma_f) \cdot \omega_1 \cdot (\mathcal{A}(\Gamma_f)^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(\mathcal{A}(\Gamma_f) \cdot \omega_2 \cdot (\mathcal{A}(\Gamma_f)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(\mathcal{A}(\Gamma_f) \cdot \omega_3 \cdot (\mathcal{A}(\Gamma_f)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(\mathcal{A}(\Gamma_f) \cdot \omega_4 \cdot (\mathcal{A}(\Gamma_f)^{-1} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(\mathcal{A}(\Gamma_f) \cdot \omega_5 \cdot (\mathcal{A}(\Gamma_f)^{-1} = \begin{pmatrix} -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -2 & -1 & 1 & 0 & 1 \end{pmatrix}$$

Now, conjugating by $(\mathcal{D}(\Gamma_f)$:

$$(\mathcal{D}(\Gamma_f) \cdot (\mathcal{A}(\Gamma_f) \cdot \omega_1 \cdot (\mathcal{A}(\Gamma_f)^{-1} \cdot (\mathcal{D}(\Gamma_f)^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(\mathcal{D}(\Gamma_f) \cdot (\mathcal{A}(\Gamma_f) \cdot \omega_2 \cdot (\mathcal{A}(\Gamma_f)^{-1} \cdot (\mathcal{D}(\Gamma_f)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(\mathcal{D}(\Gamma_f) \cdot (\mathcal{A}(\Gamma_f) \cdot \omega_3 \cdot (\mathcal{A}(\Gamma_f)^{-1} \cdot (\mathcal{D}(\Gamma_f)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(\mathcal{D}(\Gamma_f) \cdot (\mathcal{A}(\Gamma_f) \cdot \omega_4 \cdot (\mathcal{A}(\Gamma_f)^{-1} \cdot (\mathcal{D}(\Gamma_f)^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

or

$$(\mathcal{D}(\Gamma_f) \cdot (\mathcal{A}(\Gamma_f) \cdot \omega_4 \cdot (\mathcal{A}(\Gamma_f)^{-1} \cdot (\mathcal{D}(\Gamma_f)^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(\mathcal{D}(\Gamma_f) \cdot (\mathcal{A}(\Gamma_f) \cdot \omega_5 \cdot (\mathcal{A}(\Gamma_f)^{-1} \cdot (\mathcal{D}(\Gamma_f)^{-1} = \begin{pmatrix} -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 1 \end{pmatrix}$$

or

$$(\mathcal{D}(\Gamma_f) \cdot (\mathcal{A}(\Gamma_f) \cdot \omega_5 \cdot (\mathcal{A}(\Gamma_f)^{-1} \cdot (\mathcal{D}(\Gamma_f)^{-1} = \begin{pmatrix} -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -2 & -1 & -1 & 0 & 1 \end{pmatrix}$$

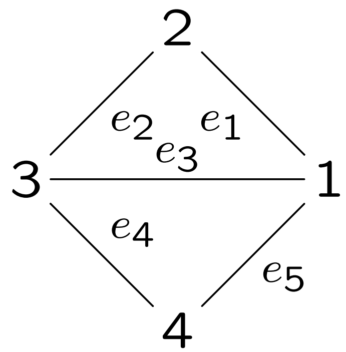
The two different choices of $(\mathcal{D}(\Gamma_f)$, gives different matrices for
$(\mathcal{D}(\Gamma_f) \cdot (\mathcal{A}(\Gamma_f) \cdot \omega_4 \cdot (\mathcal{A}(\Gamma_f)^{-1} \cdot (\mathcal{D}(\Gamma_f)^{-1}$ and for
$(\mathcal{D}(\Gamma_f) \cdot (\mathcal{A}(\Gamma_f) \cdot \omega_5 \cdot (\mathcal{A}(\Gamma_f)^{-1} \cdot (\mathcal{D}(\Gamma_f)^{-1}$, where the difference is in the last two rows of the matrices, such:

If in the first choice of $(\mathcal{D}(\Gamma_f)$, the 2×3 downer left sub-matrix is Q, then

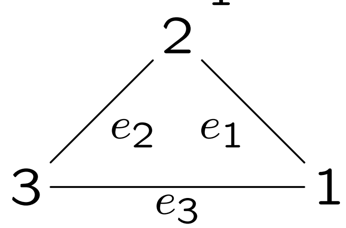
in the second choice of $(\mathcal{D}(\Gamma_f)$, the 2×3 downer left sub-matrix is $-Q$.

Example 2:

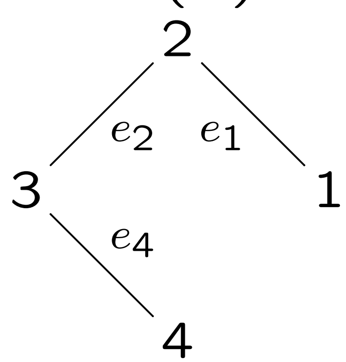
Let G be the following graph:



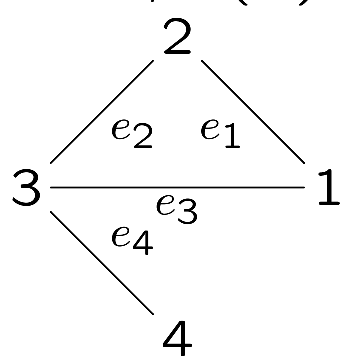
Then C_1 is:



and $T(G)$ is:



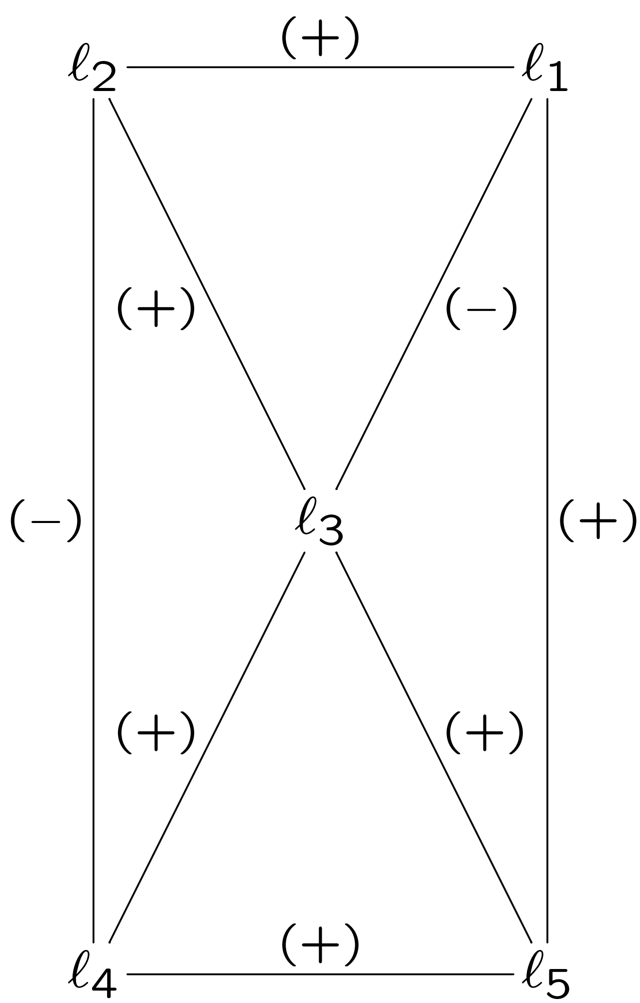
Then, $T(G) \cap C_1$ as follows:



Γ_f, where

- $f(\{\ell_1,\ell_2\}) = f(\{\ell_2,\ell_3\}) = f(\{\ell_3,\ell_4\}) = f(\{\ell_4,\ell_5\}) = f(\{\ell_3,\ell_5\}) = f(\{\ell_1,\ell_5\}) = (+1)$;

- $f(\{\ell_2,\ell_4\}) = f(\{\ell_1,\ell_3\}) = (-1)$.



Therefore:

$$\mathcal{A}(\Gamma_f) = \begin{pmatrix} 1 & 0 & 1 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Ω_{Γ_f} is generated by:

$$\omega_1 = \begin{pmatrix} -1 & 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\omega_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\omega_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\omega_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\omega_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & -1 \end{pmatrix}$$

$$\mathcal{A}(\Gamma_f) \cdot \omega_1 \cdot \mathcal{A}(\Gamma_f)^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{A}(\Gamma_f) \cdot \omega_2 \cdot \mathcal{A}(\Gamma_f)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{A}(\Gamma_f) \cdot \omega_3 \cdot \mathcal{A}(\Gamma_f)^{-1} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{A}(\Gamma_f)\cdot\omega_4\cdot\mathcal{A}(\Gamma_f)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{A}(\Gamma_f)\cdot\omega_5\cdot\mathcal{A}(\Gamma_f)^{-1} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$