The (p, q, r)-generations of the Mathieu group M_{22}

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Introduction

A finite group G is said to be (l, m, n)-*generated*, if $G = \langle x, y \rangle$, with $o(x) = l,\ o(y) = m$ and $o(xy) = o(z) = n$. Here $[x] = lX,\ [y] = mY$ and $[z] = nZ$. In this case G is also a quotient group of the triangular group $T(l, m, n)$ and, by definition of the triangular group, G is also $(\sigma(l), \sigma(m), \sigma(n))$-generated group for any $\sigma \in S_3$. Therefore we may assume that $l \leq m \leq n$. If G is a non-abelian (l, m, n)-generated group, then either $G \cong A_5$ or $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$. Thus for our purpose of establishing the (p, q, r)-generations of $G = M_{22}$, the only cases we need to consider are when $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$.

Structure Constant

- For a fixed representative $g_k \in C_k$ and for $g_i \in C_i$, $1 \leq i \leq k-1$, denote by $\Delta_G = \Delta_G(C_1, C_2, \cdots, C_k)$ the number of distinct $(k-1)$-tuples $(g_1, g_2, \cdots, g_{k-1}) \in C_1 \times C_2 \times \cdots \times C_{k-1}$ such that $g_1 g_2 \cdots g_{k-1} = g_k$. This number is known as *class algebra constant* or *structure constant*. With $\mathrm{Irr}(G) = \{\chi_1, \chi_2, \cdots, \chi_r\}$
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$$\Delta_G(C_1, C_2, \cdots, C_k) = \frac{\prod_{i=1}^{k-1} |C_i|}{|G|} \sum_{i=1}^{r} \frac{\chi_i(g_1)\chi_i(g_2)\cdots\chi_i(g_{k-1})\overline{\chi_i(g_k)}}{(\chi_i(1_G))^{k-2}} \quad (1)$$

- For a fixed $g_k \in C_k$ we denote by $\Delta^*_G(C_1, C_2, \cdots, C_k)$ the number of distinct $(k-1)$-tuples $(g_1, g_2, \cdots, g_{k-1})$ satisfying

$$g_1 g_2 \cdots g_{k-1} = g_k \quad \text{and} \quad G = \langle g_1, g_2, \cdots, g_{k-1} \rangle . \tag{2}$$

Definition

*If $\Delta^*_G(C_1, C_2, \cdots, C_k) > 0$, the group G is said to be $(C_1, C_2, \cdots, C_k)$-**generated**.*

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Subgroup Conjugates

- if H is any subgroup of G containing a fixed element $g_k \in C_k$, we let $\Sigma_H(C_1, C_2, \cdots, C_k)$ be the total number of distinct tuples $(g_1, g_2, \cdots, g_{k-1})$ such that

$$g_1 g_2 \cdots g_{k-1} = g_k \quad \text{and} \quad \langle g_1, g_2, \cdots, g_{k-1} \rangle \leq H. \qquad (3)$$

Theorem

Let $H \leq G$ containing a fixed element g such that $\gcd(o(g), [N_G(H){:}H]) = 1$. Then the number $h(g, H)$ of conjugates of H containing g is $\chi_H(g)$, where $\chi_H(g)$ is the permutation character of G with action on the conjugates of H. In particular

$$h(g, H) = \sum_{i=1}^{m} \frac{|C_G(g)|}{|C_{N_G(H)}(x_i)|},$$

where $x_1, x_2, \cdots, x_m$ are representatives of the $N_G(H)$-conjugacy classes fused to the G-class of g.

Theorems

Theorem (Ree [3])

Let G be a transitive permutation group generated by permutations $g_1, g_2, \cdots, g_s$ acting on a set of n elements such that $g_1 g_2 \cdots g_s = 1_G$. If the generator g_i has exactly c_i cycles for $1 \le i \le s$, then $\sum_{i=1}^{s} c_i \le (s-2)n + 2$.

Theorem (Scott [4])

Let $g_1, g_2, \cdots, g_s$ be elements generating a group G with $g_1 g_2 \cdots g_s = 1_G$ and $\mathbb{V}$ be an irreducible module for G with $\dim \mathbb{V} = n \ge 2$. Let $C_{\mathbb{V}}(g_i)$ denote the fixed point space of $\langle g_i \rangle$ on $\mathbb{V}$ and let d_i be the codimension of $C_{\mathbb{V}}(g_i)$ in $\mathbb{V}$. Then

$$\sum_{i=1}^{s} d_i \ge 2n.$$

Tables

Table: Cycle structures of conjugacy classes of M_{22}

nX	Cycle Structure	c_i	d_i
$1A$	1^{22}	22	0
$2A$	$1^6 2^8$	14	8
$3A$	$1^4 3^6$	10	12
$4A$	$1^2 2^2 4^4$	8	14
$4B$	$1^2 2^2 4^4$	8	14
$5A$	$1^2 5^4$	6	16
$6A$	$2^2 3^2 6^2$	6	16
$7A$	$1\ 7^3$	4	18
$8A$	$2\ 4\ 8^2$	4	18
$11A$	11^2	2	20

Table: Maximal subgroups of M_{22}

Maximal Subgroup	Orbit Lengths	Character
$L(3,4)$	[1,21]	$1a + 21a$
$2^4{:}A_6$	[6,16]	$1a + 21a + 55a$
A_7	[7,15]	$1a + 21a + 154a$
A_7	[7,15]	$1a + 21a + 154a$
$2^4{:}S_5$	[2,20]	$1a + 21a + 55a + 154a$
$2^3{:}L(3,2)$	[8,14]	$1a + 21a + 55a + 99a + 154a$
M_{10}	[10,12]	$1a + 21a + 55a + 154a + 385a$
$L(2,11)$	$[11^2]$	$1a + 21a + 55a + 154a + 210a + 231a$

Partial Fusion

Table: The partial fusion maps into M_{22}

$L(3,4)$-class	2a	3a	4a	4b	4c	5a	5b	7a	7b		
$\to$ M_{22}	2A	3A	4B	4B	4A	5A	5A	7A	7B		
h						1	1	1	1		
$2^4{:}A_6$-class	2a	2b	3a	3b	4a	4b	4c	5a	5b	6a	8a
$\to$ M_{22}	2A	2A	3A	3A	4A	4B	4A	5A	5A	6A	8A
h								1	1		
A_7-class	2a	3a	3b	4a	5a	6a	7a	7b			
$\to$ M_{22}	2A	3A	3A	4B	5A	6A	7B	7A			
h					1		1	1			
	3A	4B	5A	6A	7A	7B					
h					1		1	1			
$2^4{:}S_5$-class	2a	2b	2c	3a	4a	4b	4c	4d	5a	6a	8a
$\to$ M_{22}	2A	2A	2A	3A	4A	4B	4A	4B	5A	6A	8A
h									1		
$2^3{:}L(3,2)$-class	2a	2b	2c	3a	4a	4b	4c	6a	7a	7b	
$\to$ M_{22}	2A	2A	2A	3A	4A	4B	4A	6A	7A	7B	
h									1	1	

$(2, q, r)$ non-generations

Proposition

M_{22} is not $(2A, 3A, r)$-generated for all r.

Proof: Generally, if M_{22} is $(2A, 3A, r)$-generated group, then we must have $c_{2A} + c_{3A} + c_r \leq 24$. From Table 1 we see that $c_{2A} + c_{3A} + c_r = 24 + c_r > 24$ for all r. Now using Ree's Theorem [3], it follows that M_{22} is not $(2A, 3A, r)$-generated. ■

Proposition

M_{22} is not $(2A, 5A, 5A)$-generated group.

Proof: If M_{22} is $(2A, 5A, 5A)$-generated group, then we must have $c_{2A} + c_{5A} + c_{5A} \leq 24$. From Table 1 we see that $c_{2A} + c_{5A} + c_{5A} = 14 + 6 + 6 = 26 > 24$. Now using Ree's Theorem [3], it follows that M_{22} is not $(2A, 5A, 5A)$-generated group. ■

$(2, q, r)$ generations

Proposition

M_{22} is $(2A, 5A, 7X)$-generated group.

Proof: From Table 3 we can see that there are three maximal subgroups contain elements of orders 2, 5 and 7. These are $L(3, 4)$, A_7 and A_7. Now we have $\Delta_{M_{22}}(2A, 5A, 7A) = 224$, $\Sigma_{L(3,4)} = \Delta_{L(3,4)}(2a, 5a, 7a) + \Delta_{L(3,4)}(2a, 5b, 7a) = 63 + 63 = 126$, $\Sigma_{A_7} = \Delta_{A_7}(2a, 5a, 7a) = 14$ and $\Sigma_{A_7} = \Delta_{A_7}(2a, 5a, 7a) = 14$. The intersection of any two of the above three maximal subgroups either does not contain elements of order 2, 5 or 7. Thus there is no contribution from the intersection of any two maximal subgroups. Thus

$$\Delta^*_{M_{22}}(2A, 5A, 7A) = \Delta_{M_{22}}(2A, 5A, 7A) - \sum_{L(3,4)}(2a, 5a, 7a) - \sum_{A_7}(2a, 5$$

$$- \sum_{A_7}(2a, 5a, 7a) = 224 - 126 - 14 - 14 > 0.$$

Proposition

M_{22} is neither $(3A, 3A, 3A)$- nor $(3A, 3A, 5A)$-generated group.

Proof: The group M_{22} acts on a 21-dimensional irreducible complex module $\mathbb{V}$. By Scott's Theorem [4] applied to the module $\mathbb{V}$ and using the $\mathbb{ATLAS}$ [1], we get:

$$d_{3A} = \dim(\mathbb{V}/C_{\mathbb{V}}(3A)) = \frac{2(21-3)}{3} = 12,$$
$$d_{5A} = \dim(\mathbb{V}/C_{\mathbb{V}}(5A)) = \frac{4(21-1)}{5} = 16.$$

For the case $(3A, 3A, 3A)$ we get $d_{3A} = 3 \times 12 = 36 < 42$ and hence M_{22} is not $(3A, 3A, 3A)$-generated. For the other case $(3A, 3A, 5A)$, we get $d_{5A} = 2 \times 12 + 16 = 40 < 42$ and hence M_{22} is not $(3A, 3A, 5A)$-generated. ■

Proposition

M_{22} is $(3A, 5A, 5A)$-generated group.

Proof: Let $N \leq M_{22}$ with $N \cong L(2,11)$ and let Ω be the set of conjugates of $M' \cong L(3,4)$ in M_{22} with equal orbit lengths of 11. Then N acts on Ω. Let Γ be an orbit of length 11 with $M \in \Gamma$ and N_M be the stabilizers of M in N. Then $[N : M] = 11$ and since A_5 is the only subgroup of N of index 11 up to isomorphism, we have $N_M \cong A_5$. However

$$N_M = \{g \in N \mid M^g = M\} = \{g \in N \mid g \in N_{M_{22}}(M)\} = M = N \cap M$$

We thus have $\Delta_{M_{22}} = 2800$, $\sum_{L(3,4)} = 455$, $\sum_{L(2,11)} \leq 22$ and $\sum_{A_5} = 5$. Also if we fix an element of order 5 in N or M, then it is not contained in no other conjugate of N or M, respectively. Thus $\Delta^*_{M_{22}} = \Delta_{M_{22}} - \sum_{M \cup N_1 \cup N_2}$ where N_1 and N_2 are non-conjugate subgroups of M_{22} isomorphic to $L(2,11)$. Thus using Table 3 for fusions, we get

End of Theorem

Proof:

$$\Delta^*_{M_{22}} = 2800 - 2 \times 455 - 4 \times 22 + 4 \times 5 > 0$$

Thus M_{22} is $(3A, 5A, 5A)$-generated. ■

Theorem ([2])

Let G be a $(2X, sY, tZ)$-generated simple group, then G is $(sY, sY, (tZ)^2)$-generated.

Theorem ([2])

Let G be a finite group and let l, m and n be integers that are pairwise coprime. Then for any integer t coprime to n, we have

$$\Delta(lx, mY, nZ) = \Delta(lX, mY, (nZ)^t).$$

Moreover, G is (lX, mY, nZ)-generated if and only if G is $(lX, mY, (nZ)^t)$-generated.

More Generations

Proposition

M_{22} is $(5A, 5A, 7A)$- and $(5A, 5A, 11A)$-generated group.

Proof: This follows from the fact that M_{22} is $(2A, 5A, 7A)$- and $(2A, 5A, 11A)$-generated group, it follows by the Theorem above that M_{22} is $(5A, 5A, 7A)$- and $(5A, 5A, 11B)$-generated group. ■

Proposition

M_{22} is $(11A, 11A, 11A)$-and-$(11A, 11A, 11B)$-generated group.

Proof: This follows from the fact that M_{22} is $(2A, 11X, 11X)$ generated and so $(11X, 11X, 11X)$ generated. From the Theorem above M_{22} is $(11X, 11X, (11X)^2)$ generated. ■

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Thank You