The number of simple modules associated to Sol(q)

Jason Semeraro

Heilbronn Institute
University of Leicester
University of Bristol

8th August 2017

This is joint work with Justin Lynd.

Outline

- Blocks and fusion systems

Outline

- Blocks and fusion systems
- Alperin's weight conjecture

Outline

- Blocks and fusion systems
- Alperin's weight conjecture
- An example

Outline

- Blocks and fusion systems
- Alperin's weight conjecture
- An example
- p-local blocks and the gluing problem

Outline

- Blocks and fusion systems
- Alperin's weight conjecture
- An example
- p-local blocks and the gluing problem
- Malle–Robinson Conjecture

Outline

- Blocks and fusion systems
- Alperin's weight conjecture
- An example
- p-local blocks and the gluing problem
- Malle–Robinson Conjecture
- $\mathrm{Sol}(q)$

Blocks

- G, a finite group

Blocks

- G, a finite group
- p, a prime

Blocks

- G, a finite group
- p, a prime
- $k = \overline{k}$ with $\mathrm{char}(k) = p$

Blocks

- G, a finite group
- p, a prime
- $k = \overline{k}$ with $\mathrm{char}(k) = p$
- b, a block (indecomposable summand) of kG

Blocks

- G, a finite group
- p, a prime
- $k = \overline{k}$ with $\mathrm{char}(k) = p$
- b, a block (indecomposable summand) of kG
- M a simple kG-module $\Longrightarrow$ $Mb = b$, some b "$M \in b$"

Blocks

- G, a finite group
- p, a prime
- $k = \overline{k}$ with $\mathrm{char}(k) = p$
- b, a block (indecomposable summand) of kG
- M a simple kG-module $\Longrightarrow Mb = b$, some b "$M \in b$"
- How many such M in each b?

Fusion systems

- S, a Sylow p-subgroup of G

Fusion systems

- S, a Sylow p-subgroup of G
- $\mathcal{F} = \mathcal{F}_S(G)$ fusion system (category), objects: $\{P \leq S\}$, morphisms: G-conjugation maps

Fusion systems

- S, a Sylow p-subgroup of G
- $\mathcal{F} = \mathcal{F}_S(G)$ fusion system (category), objects: $\{P \leq S\}$, morphisms: G-conjugation maps
  - $P \leq S$ is p-centric in $G \iff Z(P) \in \mathrm{Syl}_p(C_G(P))$

Fusion systems

- S, a Sylow p-subgroup of G
- $\mathcal{F} = \mathcal{F}_S(G)$ fusion system (category), objects: $\{P \leq S\}$, morphisms: G-conjugation maps
    - $P \leq S$ is p-centric in $G \iff Z(P) \in \mathrm{Syl}_p(C_G(P))$
    - $P \leq S$ is p-radical in $G \iff O_p(N_G(P)/PC_G(P)) = 1$

Fusion systems

- S, a Sylow p-subgroup of G
- $\mathcal{F} = \mathcal{F}_S(G)$ fusion system (category), objects: $\{P \leq S\}$, morphisms: G-conjugation maps
  - $P \leq S$ is p-centric in $G \iff Z(P) \in \mathrm{Syl}_p(C_G(P))$
  - $P \leq S$ is p-radical in $G \iff O_p(N_G(P)/PC_G(P)) = 1$
- Both properties can be recovered from the category $\mathcal{F}$

Fusion systems

- S, a Sylow p-subgroup of G
- $\mathcal{F} = \mathcal{F}_S(G)$ fusion system (category), objects: $\{P \leq S\}$, morphisms: G-conjugation maps
  - $P \leq S$ is p-centric in $G \iff Z(P) \in \mathrm{Syl}_p(C_G(P))$
  - $P \leq S$ is p-radical in $G \iff O_p(N_G(P)/PC_G(P)) = 1$
- Both properties can be recovered from the category $\mathcal{F}$
- $\mathcal{F}^c := \{P \leq S \mid \ P \text{ is } p\text{-centric}\}$

Fusion systems

- S, a Sylow p-subgroup of G
- $\mathcal{F} = \mathcal{F}_S(G)$ fusion system (category), objects: $\{P \leq S\}$, morphisms: G-conjugation maps
    - $P \leq S$ is p-centric in $G \iff Z(P) \in \mathrm{Syl}_p(C_G(P))$
    - $P \leq S$ is p-radical in $G \iff O_p(N_G(P)/PC_G(P)) = 1$
- Both properties can be recovered from the category $\mathcal{F}$
- $\mathcal{F}^c := \{P \leq S \mid \ P \text{ is } p\text{-centric}\}$
- $\mathcal{F}^{cr} := \{P \leq S \mid \ P \text{ is } p\text{-centric and } p\text{-radical}\}$

Alperin's Weight Conjecture (for principal blocks)

- Suppose $k \in b$ (principal block)

Alperin's Weight Conjecture (for principal blocks)

- Suppose $k \in b$ (principal block)
- A, an algebra $\Longrightarrow z(A)$: number of projective simple A-modules

Alperin's Weight Conjecture (for principal blocks)

- Suppose $k \in b$ (principal block)
- A, an algebra $\Longrightarrow z(A)$: number of projective simple A-modules
- AWC for b (Kessar):

Conjecture (Alperin)

The number of simples in b is

$$\ell(\mathcal{F}) := \sum_{P \in \mathcal{F}^{cr}/\mathcal{F}} z(k(N_G(P)/PC_G(P)).$$

- Sum runs over a set of $\mathcal{F}$-isomorphism class representatives

An example

- Suppose $G = \mathrm{Sym}(np)$, $3 \leq p \leq n < 2p$, b principal block

An example

- Suppose $G = \mathrm{Sym}(np)$, $3 \leq p \leq n < 2p$, b principal block
- $\mathbf{p}(n) = \#$ partitions of n

An example

- Suppose $G = \mathrm{Sym}(np)$, $3 \leq p \leq n < 2p$, b principal block
- $\mathbf{p}(n) = \#$ partitions of n

Lemma

The number of simple b-modules is equal to

$$\sum_{(n)} \mathbf{p}(n_1)\mathbf{p}(n_2)\cdots\mathbf{p}(n_{p-1})$$

where the sum runs over $(n) = (n_1, \cdots, n_{p-1})$ *with* $n_j \geqslant 0$ *and* $\sum_{j=1}^{p-1} n_j = n$.

- e.g., $n = p = 3$, $(3) = (n_1, n_2) \in \{(0,3),(1,2),(2,1),(3,0)\}$ so we get $1 \cdot 3 + 1 \cdot 2 + 2 \cdot 1 + 3 \cdot 1 = 10$ simple b-modules

An example

- $S \in \mathrm{Syl}_p(G)$ has an abelian subgroup A of index p and order p^n

An example

- $S \in \mathrm{Syl}_p(G)$ has an abelian subgroup A of index p and order p^n
- $\mathcal{F}^{cr} = \{S, A\} \cup W^S$, W abelian of order p^{n-p+2}

An example

- $S \in \mathrm{Syl}_p(G)$ has an abelian subgroup A of index p and order p^n
- $\mathcal{F}^{cr} = \{S, A\} \cup W^S$, W abelian of order p^{n-p+2}
  - $P = S \Longrightarrow N_G(P)/PC_G(P) \cong C_{p-1} \wr \mathrm{Sym}(n-p) \times (C_{p-1})^2$

An example

- $S \in \mathrm{Syl}_p(G)$ has an abelian subgroup A of index p and order p^n
- $\mathcal{F}^{cr} = \{S, A\} \cup W^S$, W abelian of order p^{n-p+2}
  - $P = S \implies N_G(P)/PC_G(P) \cong C_{p-1} \wr \mathrm{Sym}(n-p) \times (C_{p-1})^2$
  - $P = W \implies N_G(P)/PC_G(P) \cong C_{p-1} \wr \mathrm{Sym}(n-p) \times \mathrm{GL}_2(p)$

An example

- $S \in \mathrm{Syl}_p(G)$ has an abelian subgroup A of index p and order p^n
- $\mathcal{F}^{cr} = \{S, A\} \cup W^S$, W abelian of order p^{n-p+2}
  - $P = S \implies N_G(P)/PC_G(P) \cong C_{p-1} \wr \mathrm{Sym}(n-p) \times (C_{p-1})^2$
  - $P = W \implies N_G(P)/PC_G(P) \cong C_{p-1} \wr \mathrm{Sym}(n-p) \times \mathrm{GL}_2(p)$
  - $P = A \implies N_G(P)/PC_G(P) \cong C_{p-1} \wr \mathrm{Sym}(n)$

An example

- $S \in \mathrm{Syl}_p(G)$ has an abelian subgroup A of index p and order p^n
- $\mathcal{F}^{cr} = \{S, A\} \cup W^S$, W abelian of order p^{n-p+2}
  - $P = S \Longrightarrow N_G(P)/PC_G(P) \cong C_{p-1} \wr \mathrm{Sym}(n-p) \times (C_{p-1})^2$
  - $P = W \Longrightarrow N_G(P)/PC_G(P) \cong C_{p-1} \wr \mathrm{Sym}(n-p) \times \mathrm{GL}_2(p)$
  - $P = A \Longrightarrow N_G(P)/PC_G(P) \cong C_{p-1} \wr \mathrm{Sym}(n)$
- Let $c_p(n)$ denote the number of p-cores of size n

Lemma

If $A = k(C_{p-1} \wr \mathrm{Sym}(n))$ *then*

$$z(A) = \sum_{(n)} c_p(n_1) \dots c_p(n_{p-1})$$

where the sum runs over $(n) = (n_1, \cdots, n_{p-1})$ *with* $n_j \geqslant 0$ *and* $\sum_{j=1}^{p-1} n_j = n$.

An example

- e.g., $n = p = 3$:

An example

- e.g., $n = p = 3$:
  - $P = S \Longrightarrow z(k(C_2 \times C_2)) = 4$

An example

- e.g., $n = p = 3$:
  - $P = S \Longrightarrow z(k(C_2 \times C_2)) = 4$
  - $P = W \Longrightarrow z(kGL_2(3)) = 2$

An example

- e.g., $n = p = 3$:
  - $P = S \Longrightarrow z(k(C_2 \times C_2)) = 4$
  - $P = W \Longrightarrow z(kGL_2(3)) = 2$
  - $P = A \Longrightarrow z(k(C_2 \wr \mathrm{Sym}(3))) = 4$

An example

- e.g., $n = p = 3$:
  - $P = S \Longrightarrow z(k(C_2 \times C_2)) = 4$
  - $P = W \Longrightarrow z(kGL_2(3)) = 2$
  - $P = A \Longrightarrow z(k(C_2 \wr \mathrm{Sym}(3)) = 4$
- So $\ell(\mathcal{F}_3(\mathrm{Sym}(9)) = 4 + 2 + 4 = 10$, as predicted by AWC!

An example

- e.g., $n = p = 3$:
  - $P = S \Longrightarrow z(k(C_2 \times C_2)) = 4$
  - $P = W \Longrightarrow z(kGL_2(3)) = 2$
  - $P = A \Longrightarrow z(k(C_2 \wr \mathrm{Sym}(3)) = 4$
- So $\ell(\mathcal{F}_3(\mathrm{Sym}(9)) = 4 + 2 + 4 = 10$, as predicted by AWC!
- In general, combine the lemmas with the identity $\mathbf{p}(n) - c_p(n) = \mathbf{p}(n - p) \cdot p$ to see that AWC holds

p-local blocks

- What about a non-principal block, b?

p-local blocks

- What about a non-principal block, b?
- $b \Longrightarrow (\mathcal{F}, \alpha)$, $\mathcal{F}$ a saturated fusion system on the defect group of b and

$$\alpha \in \lim_{[S(\mathcal{F}^c)]} \mathcal{A}^2_{\mathcal{F}}$$

  (Linckelmann) where,

p-local blocks

- What about a non-principal block, b?
- $b \Longrightarrow (\mathcal{F}, \alpha)$, $\mathcal{F}$ a saturated fusion system on the defect group of b and

$$\alpha \in \lim_{[S(\mathcal{F}^c)]} \mathcal{A}^2_{\mathcal{F}}$$

  (Linckelmann) where,

  - $[S(\mathcal{F}^c)]$: poset of $\mathcal{F}$-iso classes of chains $\sigma = (R_0 < R_1 \cdots < R_n)$ of elements in $\mathcal{F}^c$

p-local blocks

- What about a non-principal block, b?
- $b \Longrightarrow (\mathcal{F}, \alpha)$, $\mathcal{F}$ a saturated fusion system on the defect group of b and

$$\alpha \in \lim_{[S(\mathcal{F}^c)]} \mathcal{A}^2_{\mathcal{F}}$$

  (Linckelmann) where,

  - $[S(\mathcal{F}^c)]$: poset of $\mathcal{F}$-iso classes of chains $\sigma = (R_0 < R_1 \cdots < R_n)$ of elements in $\mathcal{F}^c$
  - $\mathcal{A}^2_{\mathcal{F}}$: covariant functor

$$\mathcal{A}^2_{\mathcal{F}} : \sigma \mapsto H^2(\mathrm{Aut}_{\mathcal{F}}(\sigma), k^\times)$$

p-local blocks

- What about a non-principal block, b?
- $b \Longrightarrow (\mathcal{F}, \alpha)$, $\mathcal{F}$ a saturated fusion system on the defect group of b and

$$\alpha \in \lim_{[S(\mathcal{F}^c)]} \mathcal{A}^2_{\mathcal{F}}$$

  (Linckelmann) where,

  - $[S(\mathcal{F}^c)]$: poset of $\mathcal{F}$-iso classes of chains $\sigma = (R_0 < R_1 \cdots < R_n)$ of elements in $\mathcal{F}^c$
  - $\mathcal{A}^2_{\mathcal{F}}$: covariant functor

$$\mathcal{A}^2_{\mathcal{F}} : \sigma \mapsto H^2(\mathrm{Aut}_{\mathcal{F}}(\sigma), k^\times)$$

  - $\mathrm{Aut}_{\mathcal{F}}(\sigma) \leqslant \mathrm{Aut}_{\mathcal{F}}(R_n)$: subgroup preserving R_i

- e.g. b is principal $\Longrightarrow \alpha = 0$

p-local blocks

- Call $(\mathcal{F}, \alpha)$ a *p-local block*

p-local blocks

- Call $(\mathcal{F}, \alpha)$ a *p-local block*
- If $(\mathcal{F}, \alpha)$ arises from a block b,

$$\ell(\mathcal{F}, \alpha) := \sum_{Q \in \mathcal{F}^{cr}/\mathcal{F}} z(k_{\alpha_Q} \operatorname{Out}_{\mathcal{F}}(Q)),$$

  counts the number of b-weights

p-local blocks

- Call $(\mathcal{F}, \alpha)$ a *p-local block*
- If $(\mathcal{F}, \alpha)$ arises from a block b,

$$\ell(\mathcal{F}, \alpha) := \sum_{Q \in \mathcal{F}^{cr}/\mathcal{F}} z(k_{\alpha_Q} \operatorname{Out}_{\mathcal{F}}(Q)),$$

  counts the number of b-weights
- $k_{\alpha_Q} \operatorname{Out}_{\mathcal{F}}(Q)$: algebra obtained from the group algebra $k \operatorname{Out}_{\mathcal{F}}(Q)$ by twisting with α_Q

p-local blocks

- Call $(\mathcal{F}, \alpha)$ a *p-local block*
- If $(\mathcal{F}, \alpha)$ arises from a block b,

$$\ell(\mathcal{F}, \alpha) := \sum_{Q \in \mathcal{F}^{cr}/\mathcal{F}} z(k_{\alpha_Q} \operatorname{Out}_{\mathcal{F}}(Q)),$$

  counts the number of b-weights
- $k_{\alpha_Q} \operatorname{Out}_{\mathcal{F}}(Q)$: algebra obtained from the group algebra $k \operatorname{Out}_{\mathcal{F}}(Q)$ by twisting with α_Q
- What if $(\mathcal{F}, \alpha)$ does not come from a block?

Gluing problem

- Always a natural map

$$H^2(\mathcal{F}^c, k^\times) \to \lim_{[S(\mathcal{F}^c)]} \mathcal{A}^2_{\mathcal{F}}$$

Gluing problem

- Always a natural map

$$H^2(\mathcal{F}^c, k^\times) \to \lim_{[S(\mathcal{F}^c)]} \mathcal{A}^2_{\mathcal{F}}$$

- Linckelmann asks the following in the case of blocks:

Conjecture (Gluing problem)

Is this map always surjective?

Gluing problem

- Always a natural map

$$H^2(\mathcal{F}^c, k^\times) \to \lim_{[S(\mathcal{F}^c)]} \mathcal{A}^2_{\mathcal{F}}$$

- Linckelmann asks the following in the case of blocks:

Conjecture (Gluing problem)

Is this map always surjective?

- Libman: true for all fusion systems of S_n, A_n and $\mathrm{GL}_n(q)$, $q \neq p$

Gluing problem

- Always a natural map

$$H^2(\mathcal{F}^c, k^\times) \to \lim_{[S(\mathcal{F}^c)]} \mathcal{A}^2_{\mathcal{F}}$$

- Linckelmann asks the following in the case of blocks:

Conjecture (Gluing problem)

Is this map always surjective?

- Libman: true for all fusion systems of S_n, A_n and $\mathrm{GL}_n(q)$, $q \neq p$
- Is there an exotic counterexample to the gluing problem?

Malle–Robinson conjecture

- S a p-group
- $s(S) =$ rank of largest elementary abelian subquotient (section)

Malle–Robinson conjecture

- S a p-group
- $s(S) =$ rank of largest elementary abelian subquotient (section)

Conjecture (Malle–Robinson)

 # simple b-modules $\leq p^{s(S)}$

Malle–Robinson conjecture

- S a p-group
- $s(S) =$ rank of largest elementary abelian subquotient (section)

Conjecture (Malle–Robinson)

$\#$ simple b-modules $\leq p^{s(S)}$

- AWC suggests that the following generalized version should also hold

Conjecture

Let $(\mathcal{F}, \alpha)$ be a p-local block with $\mathcal{F}$ is a saturated fusion system on S. Then $\ell(\mathcal{F}, \alpha) \leq p^{s(S)}$.

- Work of Malle–Robinson suggests that the conjecture holds for many non-exotic pairs $(\mathcal{F}, \alpha)$

Solomon–Benson fusion systems

- Goal: gather more evidence for these conjectures

Solomon–Benson fusion systems

- Goal: gather more evidence for these conjectures
- Many examples of exotic fusion systems, e.g.
  - when $p > 2$ and S has an abelian maximal subgroup (Craven–Oliver–S)

Solomon–Benson fusion systems

- Goal: gather more evidence for these conjectures
- Many examples of exotic fusion systems, e.g.
  - when $p > 2$ and S has an abelian maximal subgroup (Craven–Oliver–S)
  - when S is a Sylow 7-subgroup of the Monster (Parker–S)

Solomon–Benson fusion systems

- Goal: gather more evidence for these conjectures
- Many examples of exotic fusion systems, e.g.
  - when $p > 2$ and S has an abelian maximal subgroup (Craven–Oliver–S)
  - when S is a Sylow 7-subgroup of the Monster (Parker–S)
  - when $p = 2$: Solomon–Benson fusion systems $\mathrm{Sol}(q)$, q odd (Aschbacher–Chermak), (Levi–Oliver), (Dwyer)

Solomon–Benson fusion systems

- Goal: gather more evidence for these conjectures
- Many examples of exotic fusion systems, e.g.
  - when $p > 2$ and S has an abelian maximal subgroup (Craven–Oliver–S)
  - when S is a Sylow 7-subgroup of the Monster (Parker–S)
  - when $p = 2$: Solomon–Benson fusion systems $\mathrm{Sol}(q)$, q odd (Aschbacher–Chermak), (Levi–Oliver), (Dwyer)
- Conjecturally the Solomon–Benson systems are the only exotic simple 2-fusion systems

Solomon–Benson fusion systems

- Goal: gather more evidence for these conjectures
- Many examples of exotic fusion systems, e.g.
  - when $p > 2$ and S has an abelian maximal subgroup (Craven–Oliver–S)
  - when S is a Sylow 7-subgroup of the Monster (Parker–S)
  - when $p = 2$: Solomon–Benson fusion systems $\mathrm{Sol}(q)$, q odd (Aschbacher–Chermak), (Levi–Oliver), (Dwyer)
- Conjecturally the Solomon–Benson systems are the only exotic simple 2-fusion systems
- Morally $\mathrm{Sol}(q)$ is a group of Lie-type in char. q, a close relative of $\mathrm{Spin}_7(q)$

Solomon–Benson fusion systems

- Goal: gather more evidence for these conjectures
- Many examples of exotic fusion systems, e.g.
  - when $p > 2$ and S has an abelian maximal subgroup (Craven–Oliver–S)
  - when S is a Sylow 7-subgroup of the Monster (Parker–S)
  - when $p = 2$: Solomon–Benson fusion systems $\mathrm{Sol}(q)$, q odd (Aschbacher–Chermak), (Levi–Oliver), (Dwyer)
- Conjecturally the Solomon–Benson systems are the only exotic simple 2-fusion systems
- Morally $\mathrm{Sol}(q)$ is a group of Lie-type in char. q, a close relative of $\mathrm{Spin}_7(q)$
- Do the aforementioned representation-theoretic invariants reflect this?

…yes

- We may assume $q = 5^{2^k}$.

...yes

- We may assume $q = 5^{2^k}$.
- When $k = 0$, the elements of $\mathrm{Sol}(q)^{cr}$ are essentially given by Chermak–Oliver–Shpectorov:

P	$\lvert P \rvert$	$\mathrm{Out}_{\mathcal{F}}(P)$
S	2^{10}	1
R	2^7	A_7
R^*	2^6	S_6
RR^*	2^9	S_3
Q	2^8	$(C_3)^3 \rtimes (C_2 \times S_3)$
QR^*	2^9	S_3
QR	2^9	$(C_3 \times C_3) \overset{-1}{\rtimes} C_2$
$C_S(U)$	2^9	S_3
E	2^4	$\mathrm{GL}_4(2)$
$C_S(\Omega_1(T))$	2^7	$\mathrm{GL}_3(2)$

and when $k > 0$?

- Aschbacher–Chermak essentially list elements of $\mathrm{Sol}(q)^{cr}$ but:
  - their classification contains some errors;

and when $k > 0$?

- Aschbacher–Chermak essentially list elements of $\mathrm{Sol}(q)^{cr}$ but:
  - their classification contains some errors;
  - $\mathrm{Sol}(q)$-automorphism groups are not given explicitly

and when $k > 0$?

- Aschbacher–Chermak essentially list elements of $\mathrm{Sol}(q)^{cr}$ but:
  - their classification contains some errors;
  - $\mathrm{Sol}(q)$-automorphism groups are not given explicitly
  - the list of groups is quite different to the $k = 0$ case

and when $k > 0$?

- Aschbacher–Chermak essentially list elements of $\mathrm{Sol}(q)^{cr}$ but:
  - their classification contains some errors;
  - $\mathrm{Sol}(q)$-automorphism groups are not given explicitly
  - the list of groups is quite different to the $k = 0$ case
- Putting this all together we prove:

Theorem (Lynd-S)

Let $\mathcal{F} = \mathrm{Sol}(q)$ be a Benson-Solomon system. Then

$$\lim_{[S(\mathcal{F}^{cr})]} \mathcal{A}^2_{\mathcal{F}} \cong 0.$$

Moreover, the natural map

$$H^2(\mathcal{F}^{cr}, k^\times) \longrightarrow \lim_{[S(\mathcal{F}^{cr})]} \mathcal{A}^2_{\mathcal{F}}$$

is an isomorphism in all cases.

Calculating $\ell(\mathrm{Sol}(q), 0)$

- We can calculate the number of 'weights' for the unique p-local block $(\mathcal{F}, 0)$ supported by $\mathcal{F}$:

Theorem (Lynd-S)

For all $q > 2$, we have

$$\ell(\mathrm{Sol}(q), 0) = 12.$$

Calculating $\ell(\mathrm{Sol}(q), 0)$

- We can calculate the number of 'weights' for the unique p-local block $(\mathcal{F}, 0)$ supported by $\mathcal{F}$:

Theorem (Lynd-S)

For all $q > 2$, we have

$$\ell(\mathrm{Sol}(q), 0) = 12.$$

- Note that:
  - $\mathrm{Sol}(-)$ is behaving like a connected reductive integral group scheme $\mathbb{G}$! (assuming AWC, $\ell(\mathcal{F}_p(\mathbb{G}(q)), 0)$ is independent of q)

Calculating $\ell(\mathrm{Sol}(q), 0)$

- We can calculate the number of 'weights' for the unique p-local block $(\mathcal{F}, 0)$ supported by $\mathcal{F}$:

Theorem (Lynd-S)

For all $q > 2$, we have

$$\ell(\mathrm{Sol}(q), 0) = 12.$$

- Note that:
  - $\mathrm{Sol}(-)$ is behaving like a connected reductive integral group scheme $\mathbb{G}$! (assuming AWC, $\ell(\mathcal{F}_p(\mathbb{G}(q)), 0)$ is independent of q)
  - The generalized Malle–Robinson conjecture holds for $\mathrm{Sol}(q)$

Calculating $\ell(\mathrm{Sol}(q), 0)$

- We can calculate the number of 'weights' for the unique p-local block $(\mathcal{F}, 0)$ supported by $\mathcal{F}$:

Theorem (Lynd-S)

For all $q > 2$, we have

$$\ell(\mathrm{Sol}(q), 0) = 12.$$

- Note that:
  - $\mathrm{Sol}(-)$ is behaving like a connected reductive integral group scheme $\mathbb{G}$! (assuming AWC, $\ell(\mathcal{F}_p(\mathbb{G}(q)), 0)$ is independent of q)
  - The generalized Malle–Robinson conjecture holds for $\mathrm{Sol}(q)$
  - $\ell(\mathcal{F}_2(\mathrm{Spin}_7(q)), 0) = 12$. Is there a way to construct 'modules' for $\mathrm{Sol}(q)$ from modules in the principal 2-block of $\mathrm{Spin}_7(q)$?