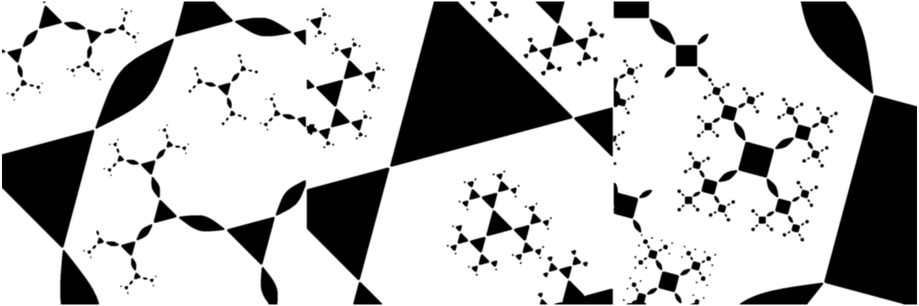
Group St Andrews 2017 in Birmingham



Another Schur-Hopf Formula

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§1 Introduction

1930s **W. Hurewicz** relates homotopy and homology groups of spaces.

1940s **S. Eilenberg** & **S. Mac Lane** introduce Homology of Groups.

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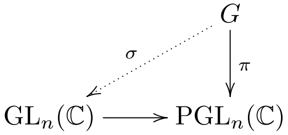
Consider the filtration $F \geq R \geq [F, F] \cap R \geq [F, R]$:

... Moreover, $R/[F, F] \cap R$ is free abelian of rank equal to the free rank of F.

In particular, the group $F/[F, R]$ is <u>infinite</u>.

§2 The Schur multiplier



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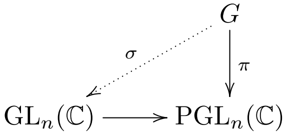

Given a finite group G with a projective representation π, any choice of the section σ determines, by mean of the relation

$$\sigma(g)\sigma(h) = \sigma(gh) \cdot \alpha(g,h) \ ,$$

a *cocycle* α in $Z^2(G) = Z^2(G, \mathbb{C}^\times)$.

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Changing the section corresponds with multiplication of α by a *coboundary*, that is an element of $B^2(G)$. Thus, one considers the *Schur multiplier*

$$M(G) = Z^2(G)/B^2(G) = H^2(G, \mathbb{C}^\times) \ .$$

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & A & \longrightarrow & \Gamma & \longrightarrow & G & \longrightarrow & 1 \\
 & & & & \vdots\ {\scriptstyle \exists ?\ \overline{\pi}} & & \downarrow {\scriptstyle \pi} & & \\
1 & \longrightarrow & \mathbb{C}^{\times} & \longrightarrow & \mathrm{GL}_n(\mathbb{C}) & \longrightarrow & \mathrm{PGL}_n(\mathbb{C}) & \longrightarrow & 1
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Given a *finite central extension* Γ one may ask, is it possible to lift π to an ordinary representation $\overline{\pi}$?

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Proof. There is a construction associating a finite subgroup H of $Z^2(G)$ with a central extension $\mathrm{SE}(H)$ of G by H (*Schur's extension*), compatibly:

$$\pi \text{ can be lifted to } \mathrm{SE}(H) \iff \text{it is represented by } \alpha \text{ in } H.$$

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... Moreover, it can be chosen in order to satisfy $A \simeq M(G)$ (*Schur cover*).

§3 The unitary cover

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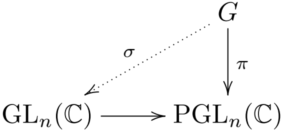
$$\sigma(g)^{o(g)} = u_\alpha(g) \cdot \mathrm{I}_n \quad , \quad u_\alpha(g) \in \mathbb{C}^\times \ .$$



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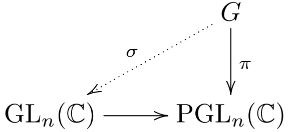

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Theorem. The *unitary cover* $\Gamma_{\mathrm{u}}(G) = \mathrm{SE}(Z_{\mathrm{u}}(G))$ is a finite extension, and it has minimal exponent among the covers of G. Also, for any $N \lhd G$, $\Gamma_{\mathrm{u}}(G)$ canonically maps onto $\Gamma_{\mathrm{u}}(G/N)$, and satisfies

$$\exp \Gamma_{\mathrm{u}}(G) \mid \exp \Gamma_{\mathrm{u}}(N) \cdot \exp \Gamma_{\mathrm{u}}(G/N)$$

Hopf's formula (modified). Given a group presentation $G \simeq F/R$, where F is free over X, let $\Omega = \langle x^{o(xR)} \mid x \in X \rangle^F$. Then

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Theorem. If G is finite, its Cayley presentation yields $\mathfrak{E}(F_c/R_c) \simeq \Gamma_{\mathrm{u}}(G)$. In addition, for any presentation, the cover satisfies $\exp \mathfrak{E}(F/R) = \exp \Gamma_{\mathrm{u}}(G)$ and, when X is finite, $\mathfrak{E}(F/R)$ is also finite.

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Finally, this offers an interpretation a result of **D.L. Johnson** (1976):

Proposition. If G is a non-cyclic p-group, then $\mathfrak{E}(F/R)$ is a proper extension.