Constructing designs invariant under the families of finite simple groups

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Preliminaries of design theory

Definition

An incidence stucture $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is a triple with point set $\mathcal{P}$, block set $\mathcal{B}$ disjoint to $\mathcal{P}$ and incidence set $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{B}$. If the ordered pair $(p, B) \in \mathcal{I}$ we say that p is incident with B.

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Definition

For a positive intger t, we say that $\mathcal{D}$ a t-design if every block $B \in \mathcal{B}$ is incident with exactly k points and every t distinct points are together incident with λ blocks. In this case we write $\mathcal{D} = t - (v, k, \lambda)$ where $v = |\mathcal{P}|$.

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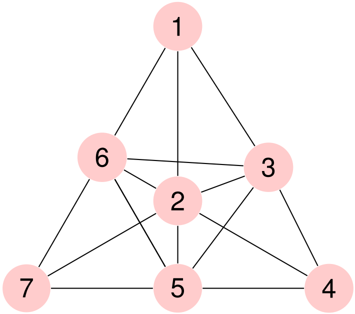
Two designs $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ and $\mathcal{D}' = (\mathcal{P}', \mathcal{B}', \mathcal{I}')$ are isomorphic if there is a bijection Φ from $\mathcal{P}$ to $\mathcal{P}'$ so that if $(p, B) \in \mathcal{I}$, then $(\Phi(p), \Phi(B) \in \mathcal{I}'$. A bijection from a design $\mathcal{D}$ to itself is called an automorphism. The group of all automorphism of $\mathcal{D}$ is denoted by $Aut(\mathcal{D})$.

Remark

Two designs with the same parameters need not to be isomprphis.

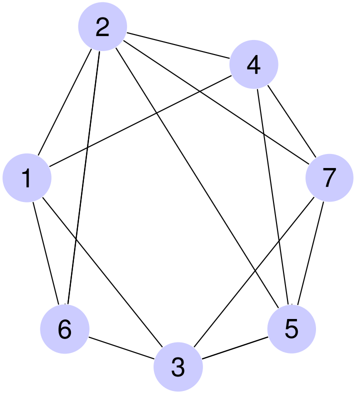
Fano plane as a $1-(7,3,3)$ design or a $2-(7,3,1)$ design

$\mathcal{B} =$
$\{\{4,5,7\},\{1,6,7\},\{1,2,5\},\{2,3,7\},\{1,3,4\},\{2,4,6\},\{3,5,6\}\}.$



A $1-(7,3,3)$ design not isomorphic to Fano plane

$$\mathcal{B} = \{\{1,2,6\},\{1,2,4\},\{2,4,7\},\{4,5,7\},\{3,5,7\},\{3,5,6\},\{1,3,6\}\}$$



Incidence matrix

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a design in which $P = \{p_1, p_2, ..., p_v\}$ and $\mathcal{B} = \{B_1, B_2, ..., B_b\}$. Then the incidence matrix of $\mathcal{D}$ is defined to be a $b \times v$ matrix $A = (a_{ij})$ such that

$$a_{ij} = \begin{cases} 1 & (p_i, B_i) \in I \\ 0 & (p_i, B_i) \notin I \end{cases}$$

Incidence matrix of a $1-(7,3,3)$ design

	$\{1,2,6\}$	$\{1,2,4\}$	$\{2,4,7\}$	$\{4,5,7\}$	$\{3,5,7\}$	$\{3,5,6\}$	$\{1,3,6\}$
1	1	1	0	0	0	0	1
2	1	1	1	0	0	0	0
3	0	0	0	0	1	1	1
4	0	1	1	1	0	0	0
5	0	0	0	1	1	1	0
6	1	0	0	0	0	1	1
7	0	0	1	1	1	0	0

Some definitions

Complement of a design

The complement of a design $\mathcal{D}$ is the structure $\mathcal{D}_1 = (\mathcal{P}, \mathcal{B}, \mathcal{I}_1)$, where $\mathcal{I}_1 = \mathcal{P} \times \mathcal{B} - \mathcal{I}$.

Dual of a design

The dual of $\mathcal{D}$ is $\mathcal{D}^t = (\mathcal{B}, \mathcal{P}, \mathcal{I}^t)$, where $(B, P) \in \mathcal{I}^t$ if and only if $(P, B) \in \mathcal{I}$. We say that a design is self-dual if it is isomorphic to its dual.

Orthoginal designs

A design is called self-orthogonal if the block intersection numbers have the same parity as the block size.

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Key-Moori Method 1

Rank of groups

Lemma

Let G be a finite simple group with a maximal subgroup M. Let $\mathcal{M}$ be the set of all congugates of M in G. Then G acts on $\mathcal{M}$ by conjugation. The point stablizer G_M of G is M, which implies that the action of G on $\mathcal{M}$ is primitive.

Definition

Consider the action of M on $\mathcal{M}$. Then the number of the orbits of this action is called the rank of G (with respect to the action of G on $\mathcal{M}$).

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Lemma

(Method 1) *Let G be a finite primitive permutation group acting on the set* Ω *of size n. Let* $\alpha \in \Omega$*, and let* $\Delta \neq \{\alpha\}$ *be an orbit of the stabilizer* G_α *of* α*. If*

$$\mathcal{B} = \{\Delta^g : \ g \in G\},$$

then $\mathcal{B}$ *forms a* 1-$(n, |\Delta|, |\Delta|)$ *design.*

Key-Moori Method 1

Our aim is to compute the parmeters of all designs constructed by Method 1, in the case that G is a simple group.

Key-Moori Method 1

Definition

Let M be a maximal subgroup of G. We define $\mathcal{A}_M = \{|M \cap M^g|| g \in G\}$.

Lemma

Let G be a finite simple group. Then the designs constructed by Method 1 are of type $1-(|G:M|, \frac{|M|}{n}, \frac{|M|}{n})$*, where* $n \in \mathcal{A}_M$*.*

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Key-Moori Method 1

So the problem of finding the parameters of designs constructed by Method 1 can be reduced to finding the set $\mathcal{A}_M$ for each maximal subgroup M of G.

Key-Moori Method 1

Lemma

Let M be a maximal subgroup of G. If the action of G on the set of maximal subgroups of G is doubly transitive then the designs constructed by Method 1 are trivial.

Reason

Since G is doubly transitive the action of M on $\mathcal{M} \setminus \{M\}$ would be transitive. Therefore $rank(G) = 2$ and the possible sizes of Δ are 1 and $n-1$.

Remark

We can apply this result to $PSL_2(q)$ (q is even), Suzuki groups $Sz(q)$ and small Ree groups $Ree(q)$.

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Results for $G = PSL_2(q)$ where q is even

Let M be a maximal subgroup of G.

- If $M \cong D_{2(q-1)}$ then $A_{\mathcal{M}} = \{1, 2, |M|\}$,
- if $M \cong D_{2(q+1)}$ then $A_{\mathcal{M}} = \{2, |M|\}$,
- if $M \cong PSL_2(q_0)$ and $q \neq q_0^2$ then

$$A_{\mathcal{M}} = \{q_0 - 1, q_0, q_0 + 1, |M|\},$$

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- if $M \cong C_{q \pm \sqrt{2q}+1}{:}4$ then $A_{\mathcal{M}} = \{1, 2, 4, |M|\}$,
- if $M \cong Sz(q_0)$ then

$$A_{\mathcal{M}} = \{1, q_0 \pm 1, q_0, q_0^2, q \pm \sqrt{2q} + 1, |M|\}.$$

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Key-Moori Method 2

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Let G be a finite simple group, M a maximal subgroup of G and let χ_M be the permutation character afforded by the action of G on $\mathcal{M}$. For $x \in M$, assume that $B = \{(M \cap x^G)^y | y \in G\}$. Then we have a $1 - (|x^G|, |M \cap x^G|, \chi_M(x))$ design $\mathcal{D}$. The group G acts as an automorphism group on $\mathcal{D}$, primitive on blocks and transitive on points of $\mathcal{D}$.

Key-Moori Method 2

Lemma

Let $\mathcal{D} = (v, k, \lambda)$ be a design obtained by the Method 2. Then $\lambda = k|G{:}M|/v$.

The first parameter of this design is well-known for most groups. So we only need to obtain either the second or the third parameter of the design.

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Remark

Let M be a maximal subgroup of G. If the action of G on the set of conjugates of M is doubly transitive then $\chi_M = 1 + \psi$, where $\psi \in Irr(G)$ and $\psi(1) = |G : M| - 1$. So if G has a unique irreducible character ψ of degree $|G : M| - 1$, then we have $\chi_M = 1 + \psi$.

Key-Moori Method 2

Remark

Let G be isomorphic to $PSL_2(q)$ (q is even). Then G has a maximal subgroup M of index $q+1$ and the action of G on conjugates of M is doubly transitive. Moreover, G has a unique irreducible character of degree q.

Key-Moori Method 2

Remark

Let G be isomorphic to $Sz(q)$. Then G has a maximal subgroup M of index $q^2 + 1$ and the action of G on conjugates of M is doubly transitive. Moreover, G has a unique irreducible character of degree q^2.

Key-Moori Method 2

Remark

Let G be isomorphic to $Ree(q)$. Then G has a maximal subgroup M of index $q^3 + 1$ and the action of G on conjugates of M is doubly transitive. Moreover, G has a unique irreducible character of degree q^3.

Definition

Let G be group and $H \leq G$. We say that H controls G-fusion in itself if each pair of elements in H are conjugate in G if and only if they are conjugate in H.

Proposition

Let G be a simple group with a maximal subgroup M and assume that M controls G-fusion in itself. Then the designs constructed by Method 2 are $1-(|x^G|, |x^M|, |C_G(x) : C_M(x)|)$ designs, where x is an element of M.

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Definition

Let G be a simple group isomorphic to either $PSL_2(q)$ (q is even) or $Sz(q)$. Let M be any maximal subgroup of G, exept the one that G acts doubly transitive on the set of it's conjugates. Then M controls G-fusion in itself.

Key-Moori Method 2

Results for $G = PSL_2(q)$ where q is even

Let M be a maximal subgroup of G of type $D_{2(q\pm 1)}$. Let $\mathcal{D}(M, x)$ be a design constructed by Method 2, the maximal subgroup M and an element $x \in M$.

- If $o(x) = 2$ then $\mathcal{D}(M, x) = (q^2 - 1, q \pm 1, q/2)$,
- if $o(x)$ divides $q - 1$ then $\mathcal{D}(M, x) = (q^2 + q, 2, 1)$,
- if $o(x)$ divides $q - 1$ then $\mathcal{D}(M, x) = (q^2 - q, 2, 1)$.

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Table 2: Designs from $Sz(q)$ using Method 2
$(q^{\pm} = q \pm \sqrt{2q} + 1)$

Max	$o(x)$ type	v	k	λ
M_1	2	$(q-1)(q^2+1)$	$q-1$	1
M_1	4	$\frac{q(q^2+1)(q-1)}{2}$	$\frac{q(q-1)}{2}$	1
M_1	$q-1$	$q^2(q^2+1)$	$2q^2$	2
M_2	2	$(q-1)(q^2+1)$	$q-1$	$\frac{q^2}{2}$
M_2	$q-1$	$q^2(q^2+1)$	2	1
M_3	2	$(q-1)(q^2+1)$	q^+	$\frac{q^2}{4}$
M_3	4	$\frac{q(q^2+1)(q-1)}{2}$	q^+	$\frac{q}{2}$
M_3	q^+	$q^2(q-1)(q^-)$	4	1
M_4	2	$(q-1)(q^2+1)$	q^-	$\frac{q^2}{4}$
M_4	4	$\frac{q(q^2+1)(q-1)}{2}$	q^-	$\frac{q}{2}$
M_4	q^-	$(q^2(q-1)(q^+)$	4	1
M_5	2	$(q^2+1)(q-1)$	$(q_0^2+1)(q_0-1)$	$\frac{q^2}{q_0^2}$
M_5	4	$\frac{q(q^2+1)(q-1)}{2}$	$\frac{q_0(q_0^2+1)(q_0-1)}{2}$	$\frac{q}{q_0}$
M_5	$q-1$	$q^2(q^2+1)$	$q_0^2(q_0^2+1)$	$\frac{q-1}{q_0-1}$
M_5	q^+	$q^2(q-1)\left(q^-\right)$	$q_0^2(q_0-1)\left(q_0^-\right)$	$\frac{q^-}{q_0^-}$
M_5	q^-	$q^2(q-1)\left(q^+\right)$	$q_0^2(q_0-1)\left(q_0^+\right)$	$\frac{q^+}{q_0^-}$

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Thank you for your attention!