Hochschild cohomology and global/local structures

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- Defect groups (p-group P of G) associated to each block. (Local structure)

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Conjecture (Auslander, Reiten)

If A and B are stably equivalent (of Morita type), then the number of isomorphism classes of non-projective simple A-modules is equal to the number of isomorphism classes of non-projective simple B-modules.

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Theorem (Koenig, Liu and Zhou (2012))

If A and B are stable equivalent of Morita type then there is an isomorphism of Lie algebras:

$$\mathrm{HH}^1(A) \cong \mathrm{HH}^1(B)$$

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Let B be a block with one simple module. Special case: nilpotent block B (Morita equivalent to kP where P defect group of B).
Second aim: If we let $\mathrm{HH}^1(B)$ be a simple Lie algebra what is the structure of the block B?

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$$\psi : \mathrm{Aut}(A[[t]]) \to \mathrm{Aut}(A[[t]]/t^r A[[t]]).$$

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Elements in $\mathrm{HD}_r(A)$ are of the form

$$\underline{D} = (\mathrm{Id}, 0 \ldots, 0, D_r, \ldots).$$

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Similarly we get a group isomorphism $\mathrm{HD}_r(A) \to \mathrm{Aut}_r(A[[t]])$.

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Proposition

The p-power map $[p]$ sends $\mathrm{HH}^1_r(A)$ to $\mathrm{HH}^1_{rp}(A)$.

Theorem (RyD)

Let A, B be finite dimensional symmetric k-algebras which are stably equivalent of Morita type. Then the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{HH}^1_r(B) & \longrightarrow & \mathrm{HH}^1_r(A) \\ \big\downarrow [p] & & \big\downarrow [p] \\ \mathrm{HH}^1_{rp}(B) & \longrightarrow & \mathrm{HH}^1_{rp}(A) \end{array}$$

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Let $A = k < x, y > / < x^p, y^p, xy + yx >$ then $\mathrm{Der}(A) = \mathrm{Der}_1(A)$.

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Proposition

Let A be as above and let B be a k-algebra stably equivalent of Morita type to A. Then:

$$\mathrm{HH}^1(B) \cong \mathrm{HH}^1(A)$$

as restricted Lie algebras.

What is known for blocks with one simple module?

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Theorem (Malle, Navarro, Späth)

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