Chief series of locally compact groups

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A **locally compact group** is a group G equipped with a locally compact Hausdorff topology, such that $(g,h) \mapsto g^{-1}h$ is continuous (where $G \times G$ carries the product topology).

A **chief factor** K/L of the locally compact group G is a pair of closed normal subgroups $L < K$ such that there are no closed normal subgroups of G lying strictly between K and L.

A **descending chief series** for G is a series of closed normal subgroups $(G_\alpha)_{\alpha \leq \beta}$ such that $G = G_0$, $1 = G_\beta$, $G_\lambda = \bigcap_{\alpha < \lambda} G_\alpha$ for each limit ordinal and each factor $G_\alpha / G_{\alpha+1}$ is chief. (Special case: finite chief series.)

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- Every finite group G has a finite chief series.
- Every **pro**finite group has a **descending** chief series with finite chief factors.
- Every connected Lie group has a finite series in which the factors are in the following list:
  1. connected centreless semisimple Lie group;
  2. finite group of prime order;
  3. $\mathbb{R}^n$, $\mathbb{Z}^n$ or $(\mathbb{R}/\mathbb{Z})^n$ for some n.

  We can also make sure all of these are chief factors except for occurrences of $\mathbb{Z}^n$ or $(\mathbb{R}/\mathbb{Z})^n$.
- Every connected locally compact group G has a descending series in which the factors come from Lie quotients of G.

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- Every connected locally compact group G has a descending series in which the factors come from Lie quotients of G.

In all the cases on the previous slide, we know what the factors can look like:

- A finite chief factor is a direct product of copies of a simple group. Finite simple groups have been classified.
- Connected centreless semisimple Lie groups are direct products of finitely many copies of an abstractly simple Lie group. Simple Lie groups have been classified.

So in some very general situations, we get a decomposition of a group G into 'known' groups. Moreover, the nonabelian chief factors we see up to isomorphism are an invariant of G (not dependent on how we constructed the series).
What about for locally compact groups that are not connected or compact?

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What about for locally compact groups that are not connected or compact?

Theorem (R.–Wesolek)

For every compactly generated locally compact group G, there is an **essentially chief series**, i.e. a finite series

$$G = G_0 > G_1 > G_2 > \cdots > G_n = \{1\}$$

of closed normal subgroups of G, such that each G_i/G_{i+1} is compact, discrete or a chief factor of G.

Theorem (R.–Wesolek)

Let G be a compactly generated locally compact group. Let $(G_i)_{i \in I}$ be a chain of closed normal subgroups of G, let $A = \overline{\bigcup_{i \in I} G_i}$ and let $B = \bigcap_{i \in I} G_i$. Then there exist $i, j \in I$ such that A/G_i and G_j/B each have a compact open G-invariant subgroup.

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From now on we assume G is a locally compact second-countable (l.c.s.c.) group. Any chief factor of G is then a *characteristically simple* l.c.s.c. group. By a result of Caprace–Monod, every compactly generated characteristically simple l.c.s.c. group is discrete, abelian or of semisimple type. However, a chief factor of a compactly generated group need not itself be compactly generated, and as a result can be much more complicated.

Wesolek (2015) defined a large class $\mathcal{E}$ of t.d.l.c.s.c. groups, the **elementary** groups, that are built from profinite and discrete groups via elementary operations. The class admits a well-behaved rank function ξ, taking values in the countable ordinals.

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Let G be a nontrivial t.d.l.c.s.c. group. Here are some characterizations of small elementary ranks.

- $\xi(G) = 2$ if and only if, for every compactly generated open subgroup H of G, then H has arbitrarily small compact open normal subgroups.
- $\xi(G) \leq n$ if and only if G has a finite normal (equivalently, characteristic) series

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such that $\xi(G_i/G_{i+1}) = 2$ for all i.
- $\xi(G) \leq \omega + 1$ if and only if, for every compactly generated open subgroup H of G, $\xi(H)$ is finite.

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Theorem (R.–Wesolek)

Let G be a nontrivial characteristically simple l.c.s.c. group. Then G has at least one of the following three structures:

(i) G is connected abelian;

(ii) (semisimple type) $G = \overline{\langle \mathcal{S} \rangle}$ where $\mathcal{S}$ is the set of topologically simple closed normal subgroups of G;

(iii) G is elementary with $\xi(G) \in \{2, \omega + 1\}$;

(iv) ('stacking phenomenon') G has a nonempty characteristic class of chief factors $\{K_i/L_i \mid i \in I\}$, such that for all $i, j \in I$, there is an automorphism α of G such that $\alpha(K_i) < L_j$ and $\alpha(C_G(K_i/L_i)) < C_G(K_j/L_j)$.

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The following indicates the potential complexity of stacking type chief factors:

Theorem (R.–Wesolek)

Let G be a t.d.l.c.s.c. group that is compactly generated and does not have arbitrarily small nontrivial compact normal subgroups. Then there is a compactly generated t.d.l.c.s.c. group E with a minimal closed normal subgroup M of stacking type, such that $\overline{[G,G]}$ is isomorphic to a subnormal factor of M.

The group E acts on a regular tree T of countably infinite degree, fixing a distinguished end of the tree, and we recover a group H with $\overline{[G,G]} \leq H \leq G$ as the action of M_v on the neighbours of a vertex v.

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Question

Does there exist an infinite sequence $G_0, G_1, G_2, \ldots$ of nonelementary (or elementary of 'high rank') characteristically simple t.d.l.c.s.c. groups, such that G_{i+1} is isomorphic to a proper chief factor of G_i for each i?

We do not know the answer even if the groups G_i are all elementary. If we did have such a sequence, then eventually $\xi(G_i) = \xi(G_{i+1}) = \xi(G_{i+2}) = \ldots,$ which appears to rule out the tree construction mentioned on the previous slide.

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Say the normal factors K_1/L_1 and K_2/L_2 are **associated** if

$$\overline{K_1 L_2} = \overline{K_2 L_1}; \; K_i \cap \overline{L_1 L_2} = L_i \text{ for } i = 1, 2.$$

E.g. for any closed normal subgroups A and B of G, $A/(A \cap B)$ is associated to $\overline{AB}/B$.

Proposition (R.–Wesolek)

For nonabelian chief factors, association is an equivalence relation. For each equivalence class, there is a canonical uppermost representative M/C, such that any chief factor associated to M/C is of the form $A/(A \cap C)$ such that $M = \overline{AC}$. In particular, there is a continuous injective homomorphism from $A/(A \cap C)$ to M/C with dense image.

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Say a chief factor is **nonnegligible** if it is nonabelian and not associated to any compact or discrete chief factor. We find that if K/L is a noncompact negligible chief factor, then $\xi(K/L) = 2$ or $K/L \cong \mathbb{R}^n$.

Theorem (R.–Wesolek)

Given an essentially chief series

$$\{1\} = G_0 < G_1 < G_2 < \cdots < G_n = G$$

for the compactly generated l.c.s.c. group G, then each association class of nonnegligible chief factors is represented exactly once as a factor G_{i+1}/G_i.

Consequently, G has only finitely many association classes of nonnegligible chief factors.

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