On Combinatorial Aspects of Abelian Groups

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Introduction

- Throughout, R is a commutative ring (with $1 \neq 0$) and all modules are unitary unless otherwise stated. A submodule N of a module M is said to be an essential submodule if it intersects non-trivially with every nonzero submodule of M. $[N : M] = \{r \in R \mid rM \subseteq N\}$ denotes an ideal of ring R.
- The ring of integers is denoted by $\mathbb{Z}$, positive integers by $\mathbb{N}$, real numbers by $\mathbb{R}$ and the ring of integers modulo n by $\mathbb{Z}_n$. Any subset of M is called an object, a combinatorial object is an object which can be put into one-to-one correspondence with a finite set of integers and an algebraic object is a combinatorial object which is also an algebraic structure.

- One of the areas in algebraic combinatorics introduced by Beck [B] is to study the interplay between graph theoretical and algebraic properties of an algebraic structure. This combinatorial approach of studying commutaive rings was explored by Anderson and Livingston in [AL]. They associated a simple graph to a commutative ring R with unity called a zero-divisor graph denoted by $\Gamma(R)$ with vertices as $Z^*(R) = Z(R)\setminus\{0\}$, where $Z(R)$ is the set of zero-divisors of R. Two distinct vertices $x, y \in Z^*(R)$ of $\Gamma(R)$ are adjacent if and only if $xy = 0$.
- The zero-divisor graph of a commutative ring has also been studied in [AFLL, SR2, RSR].

- The combinatorial properties of zero-divisors discovered in [B, AL] has also been studied in module theory. Recently in [SR1], the elements of a module M has been classified into *full-annihilators*, *semi-annihilators* and *star-annihilators*.

- Set $[x : M] = \{r \in R \mid rM \subseteq Rx\}$, an element $x \in M$ is a,
  (i) *full-annihilators*, if either $x = 0$ or $[x : M][y : M]M = 0$, for some nonzero $y \in M$ with $[y : M] \neq R$,
  (ii) *semi-annihilator*, if either $x = 0$ or $[x : M] \neq 0$ and $[x : M][y : M]M = 0$, for some nonzero $y \in M$ with $0 \neq [y : M] \neq R$,
  (iii) *star-annihilator*, if either $x = 0$ or $ann(M) \subset [x : M]$ and $[x : M][y : M]M = 0$, for some nonzero $y \in M$ with $ann(M) \subset [y : M] \neq R$.

- Denote by $A_f(M)$, $A_s(M)$ and $A_t(M)$ respectively the objects of *full-annihilators*, *semi-annihilators* and *star-annihilators*. for any module M over R and let $\widehat{A_f(M)} = A_f(M)\backslash\{0\}$, $\widehat{A_s(M)} = A_s(M)\backslash\{0\}$ and $\widehat{A_t(M)} = A_t(M)\backslash\{0\}$.

- Corresponding to *full-annihilators*, *semi-annihilators* and *star-annihilators*, the three simple graphs arising from M are denoted by $ann_f(\Gamma(M))$, $ann_s(\Gamma(M))$ and $ann_t(\Gamma(M))$ with two vertices $x, y \in M$ are adjacent if and only if $[x : M][y : M]M = 0$.

- On the other hand, the study of essential ideals in a ring R is a classical problem. For instance, Green and Van Wyk [GV] characterized essential ideals in certain class of commutative and non-commutative rings.

- The author in [A] also studied essential ideals in $C(X)$ and topologically characterized the scole and essential ideals. Moreover, essential ideals also have been investigated in C^*-algebras [KP].

- The following examples illustrate graph stuctures arising from R and M.
- **Zero-divisor graph arising from R:**

  Consider a ring $R = \mathbb{Z}_8$. We have $Z^*(\mathbb{Z}_8) = \{2, 4, 6\}$. It is easy to check that $\Gamma(\mathbb{Z}_8)$ is a path P_3 on three vertices. Similarly a zero-divisor graph $\Gamma(\mathbb{Z}_2[X, Y]/(X^2, XY, Y^2))$ arising from a ring $\mathbb{Z}_2[X, Y]/(X^2, XY, Y^2)$ is a complete graph K_3 with vertices $\{X + Y, X, Y\}$.

- **Annihilating graphs arising from M:**
  Consider a $\mathbb{Z}$-module $M = \mathbb{Z}_2 \bigoplus \mathbb{Z}_4$. Let $m_1 = (1, 0)$, $m_2 = (0, 1)$, $m_3 = (0, 2)$, $m_4 = (0, 3)$, $m_5 = (1, 1)$, $m_6 = (1, 2)$, and $m_7 = (1, 3)$ be nonzero elements of M . It can be easily verified that
  $[m_2 : M] = [m_3 : M] = [m_4 : M] = [m_5 : M] = [m_7 : M] = 2\mathbb{Z}$
  and
  $[m_1 : M] = [m_6 : M] = 4\mathbb{Z} = Ann(M)$.
  Thus, $A_f(M) = A_s(M) = \{m_1, m_2, m_3, m_4, m_5, m_6, m_7\}$ and $A_t(M) = \{m_2, m_3, m_4, m_5, m_7\}$. Since
  $[m_i : M][m_j : M]M = 0$, for all $1 \leq i, j \leq 7$, it follows that
  $ann_f(\Gamma(M)) = ann_s(\Gamma(M)) = K_7$, a complete graph on seven vertices, where as $ann_t(\Gamma(M))$ is a complete graph K_5 on five vertices.

- From the definition of annihilating graphs arising from M, the containment $ann_t(\Gamma(M)) \subseteq ann_s(\Gamma(M)) \subseteq ann_f(\Gamma(M))$ as induced subgraphs is clear, so the main emphasis is on object $\widehat{A_f(M)}$ and the full-annihilating graph $ann_f(\Gamma(M))$.

- However, one can study these objects and graphs separately for any module M.

- Following are some known results.
- **Theorem 1**, [AL]: Let R be a commutative ring with unity. Then $\Gamma(R)$ is connected and $diam(\Gamma(R)) \leq 3$. Moreover, R is finite if and only if $\Gamma(R)$ is finite.
- **Theorem 2**, [AM]: Let R and S be two finite rings which are not fields. If S is reduced and $\Gamma(R) \cong \Gamma(S)$, then $R \cong S$, unless $S \cong \mathbb{Z}_2 \times \mathbb{F}_q$, where $q = 2$ or $\frac{q+1}{2}$ is a prime power.
- More generally.
- **Theorem 3** [ALM]: Let S be a reduced ring such that S is not a domain and $\Gamma(S)$ is not a star. If R is a ring such that $\Gamma(R) \cong \Gamma(S)$, then R is a reduced ring.

- **Theorem 4** [SP1]: Let M be an R-module. Then $ann_f(\Gamma(M))$ is a connected graph and $diam(ann_f(\Gamma(M))) \leq 3$. Moreover, $ann_f(\Gamma(M))$ is finite if and only if M is finite over R .

- **Proposition 5** [SP1] Let M be a free R-module, where R is an integral domain. Then the following hold.
  (i) $ann_f(\Gamma(M))$, $ann_s(\Gamma(M))$ and $ann_t(\Gamma(M))$ are empty graphs if and only if $R \cong M$.
  (ii) $ann_s(\Gamma(M))$ and $ann_t(\Gamma(M))$ are empty graphs and the graph $ann_f(\Gamma(M))$ is complete if and only if $M \ncong R$.

- **Theorem 6** [R]: Let M and N be two R-modules such that $ann_f(\Gamma(M)) \cong ann_f(\Gamma(N))$. If $Soc(M)$ is a sum of finite simple cyclic submodules, then $Soc(M) \cong Soc(N)$.
- **Corollary 7** [R]: Let $M = \prod_{i \in I} M_i$ and $N = \prod_{i \in I} N_i$, where M_i, N_i are finite simple cyclic modules for all $i \in I$ and I is an index set. If $ann_f(\Gamma(M)) \cong ann_f(\Gamma(N))$, then $M \cong N$.
- **Corollary 8** [R]: Let M and N be two R-modules such that $ann_f(\Gamma(M)) \cong ann_f(\Gamma(N))$. If M has an essential socle, then so does N.

On Abelian Groups

- Let G be any finite $\mathbb{Z}$-module. Clearly, G is a finite abelian group. By definition of annihilating graphs, we see that there is a correspondence of ideals in R, submodules of M and the elements of objects $\widehat{A_f(M)}$, $\widehat{A_s(M)}$ and $\widehat{A_t(M)}$. Thus, we have the correspondence of ideals in $\mathbb{Z}$ and the elements of an object $\widehat{A_f(G)}$.
- Infact, the essential ideals corresponding to the submodules generated by the vertices of graph $ann_f(\Gamma(G))$ are same and the submodules determined by these vertices are isomorphic.

- For a finite abelian group $\mathbb{Z}_p \oplus \mathbb{Z}_p$, where $p \geq 2$ is prime, the essential ideals $[x : M]$, $x \in A_f(\widehat{\mathbb{Z}_p \oplus \mathbb{Z}_p})$ corresponding to the submodules of $\mathbb{Z}_p \oplus \mathbb{Z}_p$ generated by elements of $A_f(\widehat{\mathbb{Z}_p \oplus \mathbb{Z}_p})$ are same. In fact $[x : M] = ann(\mathbb{Z}_p \oplus \mathbb{Z}_p)$ for all $x \in A_f(\widehat{\mathbb{Z}_p \oplus \mathbb{Z}_p})$.

- Furthermore, the abelian group $\mathbb{Z}_p \oplus \mathbb{Z}_p$ is a vector space over field $\mathbb{Z}_p$ and all one dimensional subspaces are isomorphic. So, the submodules generated by elements of $A_f(\widehat{\mathbb{Z}_p \oplus \mathbb{Z}_p})$ are all isomorphic. For a finite abelian group $\mathbb{Z}_p \oplus \mathbb{Z}_q$, where p and q are any two prime numbers, the essential ideals determined by each $x \in A_f(\widehat{\mathbb{Z}_p \oplus \mathbb{Z}_q})$ are either $p\mathbb{Z}$ or $q\mathbb{Z}$.

- Let $\mathbb{Z}_m \otimes \mathbb{Z}_n$ be tensor product of two finite abelian groups. It is easy to verify that if g.c.d of $m, n \in \mathbb{Z}$ is 1, then $\mathbb{Z}_m \otimes \mathbb{Z}_n = \{0\}$ and in general $\mathbb{Z}_m \otimes \mathbb{Z}_n \cong \mathbb{Z}_d$, where d is g.c.d of m and n. It follows that if g.c.d of m and n is 1, then $A_f(\mathbb{Z}_m \otimes \mathbb{Z}_n) = 0$.
- However, if g.c.d of m and n is d, $d > 1$ and $\mathbb{Z}_d$ is not a simple finite abelian group, then $A_f(\mathbb{Z}_m \otimes \mathbb{Z}_n)$ contains nonzero elements, in fact the graphs $ann_f(\Gamma(\mathbb{Z}_m \otimes \mathbb{Z}_n))$ and $ann_f(\Gamma(\mathbb{Z}_d))$ are isomorphic. Furthermore, if $\mathbb{Z}_p$, $\mathbb{Z}_q$ and $\mathbb{Z}_r$ are any three finite simple abelian groups, where $p, q, r \in \mathbb{Z}$ are primes, then we have the following equality between the combinatorial objects,

$$A_f(\mathbb{Z}_p \oplus \mathbb{Z}_q \otimes \mathbb{Z}_p \oplus \mathbb{Z}_r) = A_f(\mathbb{Z}_p \oplus \mathbb{Z}_r).$$

- **Lemma 9** [R]: Let M be an R-module with $I = ann(M)$. Then $ann_f(\Gamma(M_R)) = ann_f(\Gamma(M_{R/I}))$,
  $ann_s(\Gamma(M_R)) = ann_s(\Gamma(M_{R/I}))$,
  and
  $ann_t(\Gamma(M_R)) = ann_t(\Gamma(M_{R/I}))$.

- As a consequence to Lemma 9, the annihilating graphs arising from an abelain group $\mathbb{Z}_n$ (as a $\mathbb{Z}$-module) is nothing but the zero-divisor graph of $\mathbb{Z}_n$ (as a ring).

- **Proposition 10** [R]: Let G be a finitely generated abelian group with the Betti number ≥ 2, then $ann_f(\Gamma(G))$ is complete, where the Betti number of G is the number of free factors of G.

- The following result is one of the interesting realtion between a combintorial object and an algebraic object. In this result, a combinatorial object completely determines an algebraic object. It is also a simple combinatorial characterization for non-simple finite abelian groups.

- **Proposition 11** [R]: Let G be a finite $\mathbb{Z}$-module. Then for each $x \in \widehat{A_f(G)}$, $[x : M]$ is an essential ideal if and only if G is a finite abelian group without being simple.

- **Remark 12**: Proposition 11 is not true for all $\mathbb{Z}$-modules. Consider a $\mathbb{Z}$-module $M = \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$, which is a direct sum of n copies of $\mathbb{Z}$. It is easy to verify that $\widehat{A_f(M)} = \widehat{M}$ with $[x : M][y : M]M = 0$ for all $x, y \in M$, which implies $ann_f(\Gamma(M))$ is a complete graph. The cyclic submodules generated by the vertices of $ann_f(\Gamma(M))$ are simply the lines with integral coordinates passing through the origin in the hyper plane $\mathbb{R} \oplus \mathbb{R} \oplus \cdots \oplus \mathbb{R}$ and these lines intersect at the origin only. It follows that for each $x \in M$, $[x : M]$ is not an essential ideal in $\mathbb{Z}$, in fact $[x : M]$ is a zero-ideal in $\mathbb{Z}$.

- Using the description given in Remark 12, it is now possible to characterize all the essential ideals corresponding to $\mathbb{Z}$-modules determined by elements of $\widehat{A_f(M)}$.

- **Proposition 13** [R]: If M is any $\mathbb{Z}$-module, then $[x : M]$ is an essential ideal if and only if $[x : M]$ is non-zero for all $x \in \widehat{A_f(M)}$.

- For any R-module M, it would be interesting to characterize essential ideals $[x : M]$, $x \in \widehat{A_f(M)}$ corresponding to the submodules determined by elements of $\widehat{A_f(M)}$ (or vertices of the graph $ann_f(\Gamma(M))$) such that the intersection of all essential ideals is again an essential ideal.

- It is easy to see that a finite intersection of essential ideals in any commutative ring is an essential ideal. But an infinite intersection of essential ideals need not to be an essential ideal, even a countable intersection of essential ideals in general is not an essential ideal as can be seen in [A].

- If $ann_f(\Gamma(M))$ is a finite graph, then M is a finite module over R, so the submodules determined by the vertices of graph are finite and therefore the ideals corresponding to submodules are finite in number. Therefore, it follows that the intersection of essential ideals $[x : M]$, $x \in \widehat{A_f(M)}$ in R is an essential ideal.

- Motivated by [A], I conclude with the following question regarding essential ideals corresponding to submodules M determined by vertices of the graph $ann_f(\Gamma(M))$.

- **Question:** Let M be an R-module. For $x \in \widehat{A_f(M)}$, characterize essential ideals $[x : M]$ in R such that their intersection is an essential ideal.

- This Question is true if every submodule of M is cyclic with nonzero intersection.

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Thank you for your Attention!!