Independence Complexes of Finite Groups

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Simplicial Complexes

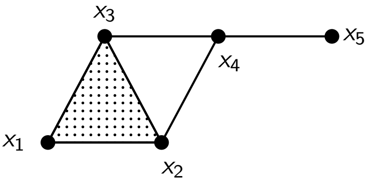
Definition

$V = \{v_1, \ldots, v_n\}$ finite set of vertices

Simplicial complex Δ on vertex set $V(\Delta)$: A collection of subsets $F \subseteq V(\Delta)$ (called **faces**) with:

- $F \in \Delta$ and $H \subseteq F \implies H \in \Delta$
- $\{v_i\} \in \Delta$ for all i.

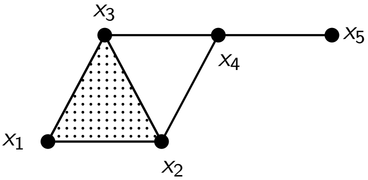
Simplicial Complexes



$$\Delta = \{\{x_1, x_2, x_3\}, \\ \{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}, \{x_2, x_4\}, \{x_3, x_4\}, \{x_4, x_5\}, \\ \{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5\}, \emptyset\}$$

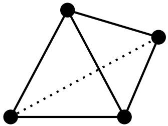
Combinatorial Information

Record the number of vertices, edges, triangles, and higher-dimensional faces

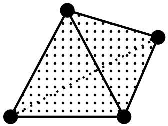


$$f_0 = 5,\ f_1 = 6,\ f_2 = 1$$

Euler Characteristic is a Topological Invariant



$$f(\Delta) = f_0 - f_1 + f_2$$
$$= 4 - 6 + 4$$
$$= 2$$



$$f(\Delta) = f_0 - f_1 + f_2 - f_3$$
$$= 4 - 6 + 4 - 1$$
$$= 1$$

Objects of Study

Definition

G finite group, non-identity elements G^*

Independent set: $S \subseteq G$, no proper subset generates the same subgroup

Fact

Independent sets of G form a simplicial complex on $V(\Delta) = G^*$

Overarching Goal

Study combinatorial properties of independent sets of finite groups via simplicial complexes

Objects of Study

First example

$C_{p_1} \times C_{p_2} \times \cdots \times C_{p_n}$ for p_i distinct primes

Goal

Count number of faces of each dimension in the simplicial complex

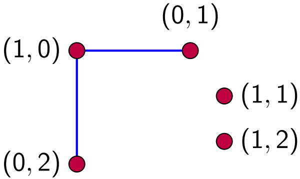
Examples

$G = C_2 \times C_3$

Independent sets of size 1: 5
$\{(1,1)\}, \{(0,2)\}, \{(1,0)\}, \{(0,1)\}, \{(1,2)\}$

Independent sets of size 2: 2
Cannot contain $(1,1)$ or $(1,2)$
(each generates whole group)
Must have form $\{(\star, 0), (0, \star)\}$
$(p_1 - 1)(p_2 - 1) = 2 \cdot 1 = 2$

Independent sets of size 3: 0
$\{(\underline{\star}, \underline{\ \ }), (\underline{\ \ }, \underline{\star}), (\underline{\ \ }, \underline{\ \ })\}$



Examples

$G = C_{p_1} \times C_{p_2} \times C_{p_3}$

Some Independent sets of size 2:
$\{(\star, 0, 0), (0, \star, 0)\}, \ldots$
$\{(\star, \star, 0), (0, 0, \star)\}, \{(\star, \star, 0), (\star, 0, \star)\}, \{(\star, \star, 0), (0, \star, \star)\},$
$\{(\star, 0, \star), (0, \star, 0)\}, \{(\star, 0, \star), (0, \star, \star)\}, \ldots$

Each tuple has a unique selling point

Counting Technique: Generalize techniques of Hearne and Wagner (*Minimal Covers of Finite Sets*) and Clarke (*Covering a Set by Subsets*)

Count the Number of Independent Sets

$n = 5, k = 3,\ A_i := p_i - 1$

$\{(\star,\star,0,0,\star),(0,0,\star,0,\star),(0,0,0,\star,0)\}$

$\downarrow$

$A_1A_2A_5|A_3A_5|A_4$

$\downarrow$

$A_1A_2|A_3|A_4$

$\downarrow$

$St(4,3) = 6$ counts the number of ways to partition $n = 4$ letters into $k = 3$ parts

$A_1A_2|A_3|A_4$

$A_1A_3|A_2|A_4$

$A_1A_4|A_2|A_3$

$A_1|A_2A_3|A_4$

$A_1|A_2A_4|A_3$

$A_1|A_2|A_3A_4$

Count the Number of Independent Sets

Each remaining non-unique variable A_j can appear in exactly

- 0 blocks in 1 way
- 2 blocks in $\binom{k}{2}$ ways, contributes $A_1 A_2 A_3 A_4 A_j^2$
- 3 blocks in $\binom{k}{3}$ ways, contributes $A_1 A_2 A_3 A_4 A_j^3$

$$\vdots$$

- k blocks in $\binom{k}{k} = 1$ way, contributes $A_1 A_2 A_3 A_4 A_j^k$

Number of Independent Sets

$G = C_{p_1} \times C_{p_2} \times \cdots \times C_{p_n}$, p_i distinct primes

Fix n, k. Let $A_i = p_i - 1$.
$St(m,k)$=number of ways to partition an m-element set into k parts

Theorem: The number of independent sets of size k in the simplicial complex for G is:

$$\sum_{m=k}^{n} \sum_{\substack{S \subseteq [n] \\ |S|=m}} St(m,k) \prod_{i \in S} A_i \prod_{j \notin S} \left(1 + \binom{k}{2} A_j^2 + \cdots + \binom{k}{k} A_j^k\right)$$

Example counts

$G = C_2 \times C_3 \times C_5 \times C_7$
$f(\Delta_G) = (1, 209, 6232, 4988, 48)$

$G = C_{11} \times C_{17} \times C_{19} \times C_{557}$
$f(\Delta_G) = (1, 1979020, 43278735636, 498994428208, 1601280)$

The End

Thank you!