On primitivity of group algebras of non-noetherian groups

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1. **Primitive group rings**

**Definition (a primitive ring)**

Let $R$ be a ring with the identity element,

$R$ is right primitive $\iff \exists M_R$ a faithful irreducible right $R$-module

$\iff \exists \rho$: a maximal right ideal of $R$ which contains no non-trivial ideals

- $R$: commutative primitive $\implies R$ is a field.
- $R$ is simple $\implies R$ is primitive.
- $R$ is artinian simple $\implies R \cong M_n(D) \cong \text{End}_D(V), \; \text{dim}_D(V) < \infty$. 

$R = \text{End}_D(V)$

$\text{dim}_D(V) = \infty$ $\implies$ $R$ is a primitive ring.
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$M$: a faithful right $R$-module:

\[ r \in R; \quad Mr=0 \Rightarrow r=0 \]

$M$: an irreducible (simple) right $R$-module:

\[ N \leq M \Rightarrow N=0 \text{ or } N=M \]
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For the case of noetherian groups

**Definition (Noetherian groups)**

A group $G$ is noetherian provided any subgroup of $G$ is finitely generated.

- $G \neq 1$: finite or abelian $\Rightarrow KG$ is never primitive.

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- $G$ is polycyclic by finite $\Rightarrow G$ is noetherian.
- $G$ is noetherian $\Rightarrow G$ is often polycyclic by finite; it is not easy to find noetherian but not polycyclic by finite.

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$KG$: primitive $\Leftrightarrow \Delta(G)=1$, $K$ is not algebraic over a finite field

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$G$ is polycyclic $\iff G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_n = 1$, $G_i / G_{i+1}$: cyclic
\( KG \) is the group algebra of a group \( G \) over a field \( K \).

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\[ \Rightarrow \Delta(G) = 1, K \text{ is not algebraic over a finite field (1979, Domanov, Farkas-Passman and Roseblade)} \]

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Δ(G): the finite conjugate center of $G$; $Δ(G)=\{g \in G \mid [G:C_G(g)]<\infty\}$
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For the case of non-noetherian groups

If $G$ is one of the following types of groups, then $KG$ is primitive for any field $K$:

- $G$ is a free product of non-trivial groups (except $G=\mathbb{Z}_2*\mathbb{Z}_2$) $\rightarrow$(1973, Formanek)

- $G$ is an amalgamated free product satisfying certain conditions $\rightarrow$(1989, Balogun)

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2. Main Results

We would like to determine the primitivity of group algebras of non-noetherian groups as generally as possible. To do this, we consider a condition satisfied by many class of groups. We first explain the notations needed.

Mutually reduced sets

Let $G$ be a group and $M$ a subset of $G$.

We denote by $\tilde{M}$ the symmetric closure of $M$; $\tilde{M} = M \cup \{x^{-1} | x \in M\}$, and by $M^x$, the set $\{x^{-1}fx | f \in M\}$, where $x \in G$.

For non-empty subsets $M_1, M_2, \ldots, M_n$ of $G$, consisting of elements $\neq 1$, we say that $M_1, M_2, \ldots, M_n$ are mutually reduced in $G$, if for each finite number of elements $g_1, g_2, \ldots, g_m \in \bigcup_{i=1}^{n} \tilde{M}_i$,

$$g_1g_2\cdots g_m = 1 \Rightarrow \exists i, j \text{ s.t. } g_i, g_{i+1} \in \tilde{M}_j.$$
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We here consider the following condition:

\[
(*) \quad \begin{cases} 
\text{For any non-empty subsets } M \text{ of } G \text{ consisting of finite number of elements } \neq 1, \\
\text{there exist } x_1, x_2, x_3 \in G \text{ such that } M^{x_1}, M^{x_2}, M^{x_3} \text{ are mutually reduced.} 
\end{cases}
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Theorem 1 ([Nishinaka and Alexander, 2017])

If $G$ is a countable infinite group and $G$ satisfies $(*)$, then $KG$ is primitive for any $K$.

This is true even if the cardinality of $G$ is general provided $G$ has a free subgroup whose cardinality is same as that of $G$ itself.
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\[ (*) \]

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Most infinite groups are non-Noetherian except for polycyclic by finite groups, and they satisfy (*).

**For example:**

- a free group,
- a free product,
- a locally free group,
- an amalgamated free product,
- an HNN-extension,
- a one relator group with torsion ...
- a non-elementary hyperbolic group

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a non-elementary hyperbolic group
Most infinite groups are non-Noetherian except for polycyclic by finite groups, and they satisfy (*).

For example:

- a free group,
- a free product,
- a locally free group,
- an amalgamated free product,
- an HNN-extension,
- a one relator group with torsion ...
- a non-elementary hyperbolic group

3. SR-graphs

We consider a Two-edge coloured graph which is simple graph (an undirected graph without loops or multi-edges).

$$V = \{v_1, v_2, ..., v_n\} \quad E = \{e_1, e_2, ..., e_m\} \quad F = \{f_1, f_2, ..., f_l\}$$

An SR-graph $S = (V, E, F)$ is an SR-graph if every component of $\mathcal{G} = (V, E)$ is a complete graph.

$I(\mathcal{G}) = \{v_3, v_6\}$
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In an SR-graph, we call an alternating cycle an SR-cycle.

\[ f_1 e_3 f_2 e_5 f_3 e_7 \]
We would like to know when an SR-graph has an SR-cycle.
Results on SR-graphs

\[ S = (V, E, F), \ \mathcal{G} = (V, E), \ \mathcal{H} = (V, F). \]

\( c(\mathcal{G}) \): the number of the set of components of \( \mathcal{G} \)
\( c(\mathcal{H}) \): the number of the set of components of \( \mathcal{H} \)

Theorem G1 ([Nishinaka and Alexander, 2017])

\( S \) is connected and each component of \( \mathcal{H} \) is complete.
Then \( S \) has an SR-cycle if and only if \( c(\mathcal{G}) + c(\mathcal{H}) < |V| + 1 \).

\( \mathcal{H}_i = (V_i, F_i) \) \((i=1,\ldots,n)\) are the components of \( \mathcal{H} \).
For \( \mathcal{H}_i \cong K_{m_1,\ldots,m_t} \), let \( \mu_i \) be \( \max\{m_1,\ldots,m_t\} \).

Theorem G2 ([Nishinaka and Alexander, 2017])
Suppose that \( \mathcal{H}_i \) is a complete multipartite graph for each \( i \).
\( |I(\mathcal{G})| \leq n \) and \( |V_i| > 2\mu_i \) for each \( i \) \( \Rightarrow \) \( S \) has an SR-cycle.
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4. An application of SR-graphs to group algebras

Let $KG$ be the group algebra of a group $G$ over a field $K$.

Let $a=\sum_{i=1}^{m} \alpha_i f_i$ and $b=\sum_{j=1}^{n} \beta_j g_j$ be in $KG$,

where $f_i, g_j \in G$ with $f_i \neq f_j, g_i \neq g_j (i \neq j)$ and $\alpha_i, \beta_j \in K \setminus \{0\}$.

Suppose $ab \in K$. Then $\sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_i \beta_j f_i g_j \in K$.

If $f_i g_j \not\in K$, $\exists k, l$, s.t. $f_i g_j = f_k g_l$.

Now, let $V = \{v_{ij} | i, j\}$ and let $E$ be the set defined by $v_{ij} v_{kl} \in E$ if $f_i g_j = f_k g_l$, and also $F$ the set done by $v_{ij} v_{st} \in F$ if $j = t$.

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Then $S = (V, E, F)$ is an SR-graph.
Suppose that there is a SR-cycle in $S$ as follows:

\[ f_1 g_1 = f_2 g_2 \\
\quad f_3 g_2 = f_4 g_3 \\
\quad f_6 g_1 = f_5 g_3 \]

\[ f_1^{-1} f_2 f_3^{-1} f_4 f_5^{-1} f_6 = 1 \]

Recall that $f_i$'s are supports of $a = \sum_{i=1}^{m} \alpha_i f_i$. So, if we prepare $f_i$'s so as not to satisfy the above equation, then we can conclude $ab \notin K$. 
Suppose that there is a SR-cycle in $S$ as follows:

\begin{align*}
v_1 & \rightarrow v_2 & f_1 g_1 &= f_2 g_2 \\
v_2 & \rightarrow v_3 & f_2 g_2 &= f_4 g_3 \\
v_3 & \rightarrow v_4 & f_3 g_2 &= f_4 g_3 \\
v_4 & \rightarrow v_5 & f_1 g_1 &= f_5 g_3 \\
v_5 & \rightarrow v_6 & f_6 g_1 &= f_5 g_3 \\
v_6 & \rightarrow v_1 & f_1^{-1} f_2 f_3^{-1} f_4 f_5^{-1} f_6 &= 1
\end{align*}

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5. How to prove primitivity of group algebras: Outline of the proof of Theorem 1

Recall:

Theorem 1 ([Nishinaka and Alexander, 2017])

If $G$ is a countable infinite group and $G$ satisfies $(\ast)$,
then $KG$ is primitive for any $K$.

where,

\[ \left\{ \begin{array}{l}
\text{For any non-empty subsets } M \text{ of } G \text{ consisting of finite number of elements } \neq 1, \\
\text{there exist } x_1, x_2, x_3 \in G \text{ such that } M^{x_1}, M^{x_2}, M^{x_3} \text{ are mutually reduced.}
\end{array} \right. \]

\[
\left\{ \begin{array}{l}
g_1, g_2, \ldots, g_m \in \bigcup_{i=1}^{3} \widetilde{M^{x_i}}, \quad g_1g_2 \cdots g_m = 1 \quad \Rightarrow \quad \exists i, j \text{ s.t. } g_i, g_{i+1} \in \widetilde{M^{x_j}}.
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\begin{align*}
  \text{g}_1, \text{g}_2, \ldots, \text{g}_m \in \bigcup_{i=1}^3 \tilde{M}^{x_i}, \text{g}_1\text{g}_2\cdots\text{g}_m = 1 & \implies \exists i, j \text{ s.t. } \text{g}_i, \text{g}_{i+1} \in \tilde{M}^{x_j}.
\end{align*}
\]
Formanek’s Method

\[ a \in KG \setminus \{0\}, \quad \varepsilon(a) \in KGaKG, \quad \rho = \sum_{a \in KG \setminus \{0\}} (\varepsilon(a) + 1)KG. \]

\[ \rho \neq KG \Rightarrow KG \text{ is primitive} \]

The main difficulty here is how to choose elements \( \varepsilon(a) \)'s so as to make \( \rho \) be proper.

Note that if \( r \in \rho \), then \( r = \sum_{t=1}^{l} (\sum_{s=1}^{3} \varepsilon(a_t) + 1) b_t \) for some \( a_t, b_t \) in \( KG \).

Let \( a_t = \sum_{i=1}^{m_t} \alpha_{ti} f_{ti} \) and \( b_t = \sum_{j=1}^{n_t} \beta_{tj} g_{tj} \) be in \( KG \).

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We can choose \( \varepsilon(a_t) \) so that \( r = \sum_{t=1}^{l} (\sum_{s=1}^{3} y_{ts} A_t + 1) b_t \).

where \( x_{ts}, y_{ts} \in G \), \( A_t = x_{t1}^{-1} a_t x_{t1} + x_{t2}^{-1} a_t x_{t2} + x_{t3}^{-1} a_t x_{t3} \).

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Formanek’s Method

\[ a \in KG\backslash \{0\}, \quad \varepsilon(a) \in KG a KG, \quad \rho = \sum_{a \in KG \backslash \{0\}} (\varepsilon(a)+1)KG. \]

\[ \rho \neq KG \Rightarrow KG \text{ is primitive} \]

The main difficulty here is how to choose elements \( \varepsilon(a) \)'s so as to make \( \rho \) be proper.

Note that if \( r \in \rho \), then \( r = \sum_{t=1}^{l} (\sum_{s=1}^{3} \varepsilon(a_t) + 1)b_t \) for some \( a_t, b_t \) in \( KG \).

Let \( a_t = \sum_{i=1}^{m_t} a_t f_{ti} \) and \( b_t = \sum_{j=1}^{n_t} b_t g_{tj} \) be in \( KG \).

where \( f_i, g_j \in G \) with \( f_i \neq f_j, g_i \neq g_j \) (\( i \neq j \)) and \( \alpha_i, \beta_j \in K \backslash \{0\} \).

We can choose \( \varepsilon(a_t) \) so that \( r = \sum_{t=1}^{l} (\sum_{s=1}^{3} y_{ts} A_t + 1) b_t \).

where \( x_{ts}, y_{ts} \in G \), \( A_t = x_{t1}^{-1} a_t x_{t1} + x_{t2}^{-1} a_t x_{t2} + x_{t3}^{-1} a_t x_{t3} \).

All we have to do is to show, \( r = \sum_{t=1}^{l} (\sum_{s=1}^{3} y_{ts} A_t + 1) b_t \neq 1 \).
If \( M^{st} = \{ x_{st}^{-1} f_{t1} x_{st}, \ldots, x_{st}^{-1} f_{tm} x_{st} \} \) (s = 1,2,3) are mutually reduced and \( y_{ts} \) (1 \( \leq t \leq l, 1 \leq s \leq 3 \)) are also mutually reduced, then we have

\[
r = \sum_{t=1}^{l} \left( \sum_{s=1}^{3} y_{ts} A_t + 1 \right) b_t \neq 1.
\]

In fact, suppose, to the contrary, that \( r = 1 \).

\[
r = \sum_{t,s=1}^{l,3} (y_{ts} A_t + 1) b_t = \sum_{s=1}^{3} (y_{1s} A_1 b_1 + b_1) + \ldots + \sum_{s=1}^{3} (y_{ts} A_t b_t + b_t) + \ldots + \sum_{s=1}^{3} (y_{ls} A_l b_l + b_l) = 1.
\]

By Theorem G2, \( |\text{Supp}(A_t b_t)| > n_t \).

By this result and Theorem G1 implies \( y_{is}^{-1} y_{jt} \ldots y_{kp}^{-1} y_{lj} = 1 \)

for \((i,s) \neq (j,t), \ldots, (k,p) \neq (l,q)\); a contradiction.

Recall:

\[
A_t b_t = x_{t1}^{-1} a_t x_{t1} + x_{t2}^{-1} a_t x_{t2} + x_{t3}^{-1} a_t x_{t3},
\]

\[
a_t = \sum_{i=1}^{m_t} a_t f_{ti} \quad \text{and} \quad b_t = \sum_{j=1}^{n_t} \beta_{tj} g_{tj}.
\]
If $M^{x_{st}} = \{ x_{st}^{-1} f_{t1} x_{st}, \ldots, x_{st}^{-1} f_{tm_t} x_{st} \} (s = 1, 2, 3)$ are mutually reduced and $y_{ts} \ (1 \leq t \leq l, 1 \leq s \leq 3)$ are also mutually reduced, then we have

$$r = \sum_{t=1}^{l} (\sum_{s=1}^{3} y_{ts} A_t + 1)b_t \neq 1.$$

In fact, suppose, to the contrary, that $r = 1$.

$$r = \sum_{t, s = 1}^{l, 3} (y_{ts} A_t + 1)b_t = \sum_{s = 1}^{3} (y_{1s} A_1 b_1 + b_1) + \cdots + \sum_{s = 1}^{3} (y_{ts} A_t b_t + b_t) + \cdots + \sum_{s = 1}^{3} (y_{ls} A_l b_l + b_l) = 1.$$

By Theorem G2, $|\text{Supp}(A_t b_t)| > n_t$.

By this result and Theorem G1 implies $y_{ls}^{-1} y_{jt} \cdots y_{kp}^{-1} y_{lq} = 1$ for $(i, s) \neq (j, t), \ldots, (k, p) \neq (l, q)$; a contradiction.

Recall:

$$A_t b_t = x_{t1}^{-1} a_t x_{t1} + x_{t2}^{-1} a_t x_{t2} + x_{t3}^{-1} a_t x_{t3},$$

$$a_t = \sum_{i=1}^{m_t} a_{ti} f_{ti} \text{ and } b_t = \sum_{j=1}^{n_t} \beta_{tj} g_{tj}.$$
If $M^{x_{st}} = \{x_{st}^{-1} f_{t1} x_{st}, \ldots, x_{st}^{-1} f_{tm_{t}} x_{st}\} (s = 1, 2, 3)$ are mutually reduced and $y_{ts} (1 \leq t \leq l, 1 \leq s \leq 3)$ are also mutually reduced, then we have

$$r = \sum_{t=1}^{l} (\sum_{s=1}^{3} y_{ts} A_t + 1)b_t \neq 1.$$ 

In fact, suppose, to the contrary, that $r = 1$.

$$r = \sum_{t,s=1}^{l,3} (y_{ts} A_t + 1)b_t = \sum_{s=1}^{3} (y_{1s} A_1 b_1 + b_1) + \cdots + \sum_{s=1}^{3} (y_{ts} A_t b_t + b_t) + \cdots + \sum_{s=1}^{3} (y_{ls} A_l b_l + b_l) = 1.$$ 

By Theorem G2, $|\text{Supp}(A_t b_t)| > n_t$.

By this result and Theorem G1 implies

$$y_{is}^{-1} y_{jt} \cdots y_{kp}^{-1} y_{lq} = 1$$

for $(i, s) \neq (j, t), \ldots, (k, p) \neq (l, q)$; a contradiction.

Recall:

$$A_t b_t = x_{t1}^{-1} a_t x_{t1} + x_{t2}^{-1} a_t x_{t2} + x_{t3}^{-1} a_t x_{t3},$$

$$a_t = \sum_{i=1}^{m_t} a_{ti} f_{ti} \quad \text{and} \quad b_t = \sum_{j=1}^{n_t} \beta_{tj} g_{tj}.$$
If \( M^{x_{st}} = \{ x_{st}^{-1}f_{t_{1}}x_{st}, \ldots, x_{st}^{-1}f_{t_{m_{t}}}x_{st} \} \) \((s = 1,2,3)\) are mutually reduced and \( y_{ts} \) \((1 \leq t \leq l, 1 \leq s \leq 3)\) are also mutually reduced, then we have

\[
r = \sum_{t=1}^{l}(\sum_{s=1}^{3}y_{ts}A_{t} + 1)b_{t} \neq 1.
\]

In fact, suppose, to the contrary, that \( r = 1 \).

\[
r = \sum_{t,s=1}^{l,3}(y_{ts}A_{t} + 1)b_{t} = \sum_{s=1}^{3}(y_{1s}A_{1}b_{1} + b_{1}) + \cdots + \sum_{s=1}^{3}(y_{ts}A_{t}b_{t} + b_{t}) + \cdots + \sum_{s=1}^{3}(y_{ls}A_{l}b_{l} + b_{l}) = 1.
\]

By Theorem G2, \(|\text{Supp}(A_{t} b_{t})| > n_{t} .\)

By this result and Theorem G1 implies \( y_{is}^{-1}y_{jt} \cdots y_{kp}^{-1}y_{lq} = 1 \) for \((i, s) \neq (j, t), \cdots, (k, p) \neq (l, q); \)
a contradiction.

Recall:

\[
\begin{align*}
A_{t} b_{t} &= x_{t_{1}}^{-1}a_{t}x_{t_{1}} + x_{t_{2}}^{-1}a_{t}x_{t_{2}} + x_{t_{3}}^{-1}a_{t}x_{t_{3}}, \\
\sigma_{t} &= \sum_{i=1}^{m_{t}}a_{ti}f_{ti} \quad \text{and} \quad b_{t} = \sum_{j=1}^{n_{t}}\beta_{tj}g_{tj}.
\end{align*}
\]
If $M^{x_{st}} = \{ x_{st}^{-1} f_{t_1} x_{st}, \ldots, x_{st}^{-1} f_{m_t} x_{st}\} (s = 1,2,3)$ are mutually reduced and $y_{ts} (1 \leq t \leq l, 1 \leq s \leq 3)$ are also mutually reduced, then we have

$$r = \sum_{t=1}^{l} (\sum_{s=1}^{3} y_{ts} A_t + 1)b_t \neq 1.$$ 

In fact, suppose, to the contrary, that $r = 1$.

$$r = \sum_{t,s=1}^{l,3} (y_{ts} A_t + 1)b_t = \sum_{s=1}^{3} (y_{1s} A_1 b_1 + b_1) + \cdots + \sum_{s=1}^{3} (y_{ts} A_t b_t + b_t) + \cdots + \sum_{s=1}^{3} (y_{ls} A_l b_l + b_l) = 1.$$ 

By Theorem G2, $|\text{Supp}(A_t b_t)| > n_t$.

By this result and Theorem G1 implies $y_{is}^{-1} y_{jt} \cdots y_{kp}^{-1} y_{lq} = 1$ for $(i,s) \neq (j,t), \cdots, (k,p) \neq (l,q)$; a contradiction.

Recall:
$$A_t \ b_t = x_{t_1}^{-1} a_t x_{t_1} + x_{t_2}^{-1} a_t x_{t_2} + x_{t_3}^{-1} a_t x_{t_3},$$
$$a_t = \sum_{i=1}^{m_t} a_{ti} f_{ti} \text{ and } b_t = \sum_{j=1}^{n_t} \beta_{tj} g_{tj}.$$
If $M^{x_{st}} = \{ x_{st}^{-1}f_{t1}x_{st}, \ldots, x_{st}^{-1}f_{tm_t}x_{st} \}$ $(s = 1,2,3)$ are mutually reduced and $y_{ts}$ $(1 \leq t \leq l, 1 \leq s \leq 3)$ are also mutually reduced, then we have

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In fact, suppose, to the contrary, that $r = 1$.

$$r = \sum_{t,s=1}^{l,3} (y_{ts}A_t + 1)b_t = \sum_{s=1}^{3} (y_{1s}A_1b_1 + b_1) + \cdots + \sum_{s=1}^{3} (y_{ts}A_tb_t + b_t) + \cdots + \sum_{s=1}^{3} (y_{ls}A_l b_l + b_l) = 1.$$ 

By Theorem G2, $|\text{Supp}(A_t b_t)| > n_t$.

By this result and Theorem G1 implies $y_{is}^{-1}y_{jt} \cdots y_{kp}^{-1}y_{lq} = 1$ for $(i,s) \neq (j,t), \cdots, (k,p) \neq (l,q)$; a contradiction.

Recall:

$$\begin{aligned}
A_t b_t &= x_{t1}^{-1}a_t x_{t1} + x_{t2}^{-1}a_t x_{t2} + x_{t3}^{-1}a_t x_{t3}, \\
a_t &= \sum_{i=1}^{m_t} a_{ti}f_{ti} \quad \text{and} \quad b_t = \sum_{j=1}^{n_t} \beta_{tj}g_{tj}.
\end{aligned}$$
Thank you!

[N, 2016] “Uncountable locally free groups and their group rings”
arXiv:1601.00295

[N and A, 2017] “Non-noetherian groups and primitivity of their group algebras”
J. Algebra Vol. 473

[N, 2011] “Group rings of countable non-abelian locally free groups are primitive”
Int. J. alg. and comp Vol 21

[N,2007] “Group rings of proper ascending HNN extensions of countably infinite free groups are primitive”
J. Algebra Vol. 317