On $p$-groups of conjugate rank 1 and nilpotency class 3.

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(Joint work with Rahul Kitture and Manoj Yadav)

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Theorem (J. Cossey-T. O. Hawkes, 2000)
Let $p$ be a prime and $0 = e_0 < e_1 < \cdots < e_n$ be integers. Then there exists a $p$-group $G$ with nilpotency class 2 such that, the set of conjugacy class sizes of $G$ is exactly
\[ \{1 = p^{e_0}, p^{e_1}, \ldots, p^{e_n} \}. \]
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Problem 1 (Avinoam Mann, 2011)
Find other constructions, in particular ones that produce groups of higher class.

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**Problem 1 (Avinoam Mann, 2011)**

Find other constructions, in particular ones that produce groups of higher class.

**Problem 2**

What about groups with exactly two conjugacy class sizes?
Let’s go in history
A finite group $G$ is said to be of *conjugate type* $(1 = m_0, m_1, \ldots, m_r)$; if $m_i$’s are precisely the different sizes of conjugacy classes of $G$. Here we say that $G$ is of conjugate rank $r$. 

In the 1953, N. Ito started the study of finite groups with few conjugacy class sizes. In a series of paper "On finite groups with given conjugate type I, II, III (1953, 1970, 1970)", he studied finite groups with 2, 3, 4 conjugacy class sizes respectively. In this talk, we concentrate mainly on finite groups with exactly two conjugacy class sizes.
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In this talk, we concentrate mainly on finite groups with exactly two conjugacy class sizes.
Theorem (N. Ito, 1953)

Let $G$ be a finite group with exactly two conjugacy class sizes, namely 1 and $m$. Then the following hold;

$m$ is a power of some prime $p$, say $m = p^n$.

$G = P \times A$, where $P$ is the non-abelian Sylow $p$-subgroup of $G$ and $A$ is an abelian $p'$-subgroup of $G$.

In particular, $G$ is nilpotent.
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Problem 4

Classify finite $p$-groups of conjugate type $(1, p^n)$, where $n \geq 1$. 
Theorem (I.M. Isaacs, 1970)

Let $G$ be a finite group, which contain a proper normal subgroup $N$ such that all the conjugacy classes of $G$, which lie outside $N$ have same lengths.

Corollary 1

Let $G$ be a finite $p$-group with conjugate type $(1, p^n)$. Then $\exp(G/Z(G)) = p$.

Corollary 2

Let $G$ be a finite 2-group with conjugate type $(1, 2^n)$. Then the nilpotency class of $G$ is exactly 2.
Theorem (I.M. Isaacs, 1970)

Let $G$ be a finite group, which contain a proper normal subgroup $N$ such that all the conjugacy classes of $G$, which lie outside $N$ have same lengths. Then either $G/N$ is cyclic or every non-identity element of $G/N$ is of prime order.

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Let $p$ be an odd prime and $G$ be a $p$-group with exactly two conjugacy class sizes. Then the nilpotency class of $G$ is either 2 or 3. Mann and Isaacs independently generalized this.
Now we summarize the situation on Problem 1 and Problem 2, (with the extra conditions) for the groups having exactly two conjugacy class sizes.

1. Given any odd prime $p$ and any integer $n \geq 1$, we cannot construct finite $p$-group of conjugate type $(1, p^n)$, with nilpotency class greater than 3.

2. Given any integer $n \geq 1$, we cannot construct finite 2-group of conjugate type $(1, 2^n)$, with nilpotency class greater than 2.

Now, we concentrate on Problem 4; Problem 4

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**Isoclinism (P. Hall, 1940)**

Two finite groups $G$ and $H$ are called *isoclinic* if there exists an isomorphism $\phi$ of the factor group $\bar{G} = G/\mathbb{Z}(G)$ onto $\bar{H} = H/\mathbb{Z}(H)$, and an isomorphism $\theta$ of the subgroup $G'$ onto $H'$ such that the following diagram is commutative

\[
\begin{array}{ccc}
\bar{G} \times \bar{G} & \xrightarrow{a_G} & G' \\
\phi \times \phi \downarrow & & \downarrow \theta \\
\bar{H} \times \bar{H} & \xrightarrow{a_H} & H'.
\end{array}
\]

$a_G$ and $a_H$ are canonical commutator maps.
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Before going to the classification, we exhibit some examples.
For any positive integer \( r \geq 1 \) and prime \( p > 2 \), consider the following group constructed by N. Ito.

\[
G_r = \left\langle a_1, \ldots, a_{r+1} \mid [a_i, a_j] = b_{ij}, [a_k, b_{ij}] = 1, \right.
\]
\[
a_i^p = a_{r+1}^p = b_{ij}^p = 1, 1 \leq i < j \leq r + 1, 1 \leq k \leq r + 1 \right\rangle.
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The group $G_r$ is of conjugate type $(1, p^r)$ and nilpotency class 2.
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The group \( G_r \) is of conjugate type \( (1, p^r) \) and nilpotency class 2.

For any \( k \geq 1 \), the group \( U_3(p^n) \) of upper unitriangular matrices over a field of order \( p^n \) is of conjugate type \( (1, p^n) \) and class 2.
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The family of Camina special \( p \)-groups \( G \), with \( |G'| = p^k \) provides a huge source of examples of groups of conjugate type \( (1, p^k) \) and class 2.
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These examples were appeared in the construction of certain Camina $p$-groups of class 3 by Dark and Scoppola in 1996.

It can be showed that; for fix $n$, the $p$-group of conjugate type $(1, p^{2n})$ and class 3, constructed by Dark and Scoppola is isomorphic to $\mathcal{H}_n/Z(\mathcal{H}_n)$, where $\mathcal{H}_n$ can be presented as below;

\[
\mathcal{H}_n = \left\{ \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
a & 1 & 0 & 0 & 0 \\
c & b & 1 & 0 & 0 \\
d & ab - c & a & 1 & 0 \\
f & e & c & b & 1
\end{bmatrix} : a, b, c, d, e, f \in \mathbb{F}_{p^n} \right\}.
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Now, we come to the classification of $p$-groups having exactly 2 class sizes.
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**Theorem (K. Ishikawa, 1999)**
A finite $p$-group $G$ has exactly two conjugacy class sizes 1 and $p$ if and only if $G$ is isoclinic to an **extra special $p$-Group**.
Theorem (K. Ishikawa, 1999)

- Let \( G \) be a finite \( p \)-group of conjugate type \((1, p^2)\) and nilpotency class 2. Then \( G \) is isoclinic to one of the following:

1. A Camina group \( H \) with \(|H'| = p^2\).
2. \( G_{2} = \langle a_1, a_2, a_3 | [a_i, a_j] = b_{ij}, [a_k, b_{ij}] = 1, a_{p^i} = b_{p^i} = c_{p^i} = 1, 1 \leq i < j \leq 3, 1 \leq k \leq 3 \rangle \).

Let \( G \) be a finite \( p \)-group of conjugate type \((1, p^2)\) and nilpotency class 3. Then \( G \) is isoclinic to \( W \), where \( W \) can be presented as,

\[ W = \langle a_1, a_2 | [a_1, a_2] = b, [a_i, b] = c_i, a_{p^i} = b_{p^i} = c_{p^i} = 1, i = 1, 2 \rangle. \]

Note that \( W \) is isomorphic to the group constructed by Dark and Scoppola, \( H_n/Z(H_n) \); for \( n = 1 \).
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2. $G_r$, for $r = 2$.

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Note that $W$ is isomorphic to the the group constructed by Dark and Scoppola, $\mathcal{H}_n/Z(\mathcal{H}_n)$; for $n = 1$. 

Theorem(Tushar K. Naik, Manoj K. Yadav (2017))

Let $G$ be a finite $p$-group of conjugate type $\{1, p^3\}$, $p > 2$. Then nilpotency class of $G$ is 2.
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- A finite Camina $p$-group of nilpotency class 2 with commutator subgroup of order $p^3$;
- The group $G_r$, for $r = 3$;

$$G_3 = \left\langle a_1, \ldots, a_4 \mid [a_i, a_j] = b_{ij}, [a_k, b_{ij}] = 1, a_i^p = a_4^p = b_{ij}^p = 1, 1 \leq i < j \leq 4, 1 \leq k \leq 4 \right\rangle.$$
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- The quotient group $G_3/M$, where $M$ is a normal subgroup of $G_3$ given by $M = \langle [a_1, a_2][a_3, a_4] \rangle$;
Theorem (Tushar K. Naik, Manoj K. Yadav (2017))

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- The quotient group \( G_3/N \), where \( N \) is a normal subgroup of \( G_3 \) given by \( N = \langle [a_1, a_2][a_3, a_4], [a_1, a_3][a_2, a_4]^t \rangle \), with \( t \) any fixed integer non-square modulo \( p \).
Let $\hat{G}_n$ denote the family consisting of $(n + 1)$-generator non-abelian special $p$-groups $G$ of order $p^{(n+1)(n+2)/2}$. Let $\hat{G}_3$ denote the subfamily of $\hat{G}_3$ consisting of 2-groups of exponent 4.
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Then it follows that all groups of this family are of conjugate type $\{1, 2^3\}$. It also turns out that any two groups in $\hat{G}_3$ are isoclinic.
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Then it follows that all groups of this family are of conjugate type $\{1, 2^3\}$. It also turns out that any two groups in $\hat{G}_3$ are isoclinic.

For simplicity of notation, we assume that a group $G$ from $\hat{G}_3$ is minimally generated by the set $\{a, b, c, d\}$. 
Theorem (Tushar K. Naik, Manoj K. Yadav, 2017)
Let $G$ be a finite 2-group of conjugate type $\{1, 8\}$ and nilpotency class 2. Then $G$ is isoclinic to one of the following groups:

(i) A finite Camina 2-group with commutator subgroup of order 8;
(ii) A fixed group $G$ in the family $\hat{G}_3$, defined above;
(iii) The quotient group $G/M$, where $M$ is a normal subgroup of $G$ such that $M = \langle [a, b] [c, d] \rangle$;
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All these information lead to the following natural questions.

**Question 6**
Does there exist a finite $p$-group of nilpotency class 3 and conjugate type $(1, p^n)$, for odd prime $p$ and odd integer $n \geq 5$?

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For given even integer $n$, does there exist more groups of conjugate type $(1, p^n)$, other than the example constructed by Dark and Scoppola?
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Let $p$ be an odd prime. Then the following holds;

There does not exist any $p$-group of conjugate type $\{1, p^n\}$ and nilpotency class 3, for odd integer $n$.

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Thank You