On p-groups with automorphism groups of prescribed properties

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- $\varphi : \mathrm{Aut}(P) \to \mathrm{Aut}(P/\Phi(P)) \to \mathrm{GL}(d, p)$, and let $A(P) := \varphi(\mathrm{Aut}(P))$.
- Which groups occur as $A(P)$ for some p-group P?

Given H, do there exist p-groups P such that $A(P) \cong H$?

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Linear "representation"

Amongst such P, what is the minimal

- order,
- exponent,
- nilpotency class?

Inducing groups on the central quotient

Theorem (Heineken, Liebeck)

Let H be a group and p an odd prime. There exists a p-group P of exponent p^2 and nilpotency class two such that the group induced on $P/Z(P)$ is isomorphic to H.

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Rank of P is $\approx |H|$.

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Given $H \leqslant GL(d,p)$*, there exists a* d*-generator* p*-group* P *such that* $A(P) = H$*.*

This is a linear “representation”.

There is no bound on nilpotency class, exponent or order.

Rarity of such p-groups

Theorem (Helleloid, Martin)

Let $d \geqslant 5$.

$$\lim_{n\to\infty} \begin{pmatrix} \textit{proportion of } d\textit{-generator } p\textit{-groups} \\ \textit{with } p\textit{-length at most } n \\ \textit{with automorphism group a } p\textit{-group} \end{pmatrix} = 1.$$

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"The automorphism group of a p-group is almost always a p-group."

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Amongst such P, minimise:

- exponent – aim for exponent p ?
- nilpotency class – aim for class $\leqslant 3$?
- order.

Where to look?

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If X is a p-group of exponent p: $\lambda_i(X)/\lambda_{i+1}(X)$ is an elementary abelian p-group.

Where to look

Let $d \geqslant 2$ and $n \geqslant 1$ be integers.

Set $B(d,p) = F_d/(F_d)^p$, the relatively free group of rank d and exponent p.

Set $\Gamma(d,n) = B(d,p) \,/\, \lambda_n(B(d,p))$.

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$\Gamma(d,n)$ is the relatively free d-generator group of exponent p and class n.

Properties of $\Gamma(d, n)$

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- $\Gamma(d, n)$ is a finite p-group (order formula due to Witt).
- $A(\Gamma(d, n)) = GL(d, p)$ (as large as possible)
- If P is a finite d-generator p-group of exponent p and class at most n, then P is a quotient of $\Gamma(d, n)$.

Automorphisms of quotients

$\Gamma(d, n)$ is the (relatively free) d-generator group of exponent p and class n.

Let

$$U < \lambda_{n-1}(\Gamma(d, n))$$

and set

- $H := N_{GL(d,p)}(U), \quad (= N_{A(\Gamma(d,n))}(U))$
- $P := \Gamma(d, n)/U.$

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- $H := N_{GL(d,p)}(U), \quad (= N_{A(\Gamma(d,n))}(U))$
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Then P is a d-generator, exponent p, class n finite p-group with

$$A(P) = H.$$

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Problem is to show $H = N_{GL(d,p)}(U)$.
When H is maximal – this is easy.

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$$\Gamma(d, 2) = \{(u, v) \mid u \in V,\ v \in A^2(V)\}$$

Multiplication: $(u, v)(u', v') = (u + u', v + v' + u \wedge v)$.

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- For H stabiliser of proper non-trivial subspace, not as in (*), there exists a d-generator p-group P of exponent p and class 2 with $A(P) = H$.
- For H as in (*), there is no class 2 group P with $A(P) = H$.

Another example

H preserves an alternating form β (up to scalars)

$$\pi : A^2(V) \to \mathbb{F}_p, \quad \pi : u \wedge v \mapsto \beta(u, v)$$

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$\Gamma(d, 2)/\ker \pi \cong p_+^{1+\dim V}$.

Main Result

Theorem (Bamberg, Glasby, M., Niemeyer)

Let $p \geqslant 5$ be a prime, and let $d \geqslant 2$ be an integer. Suppose that H is a maximal subgroup of $\mathrm{GL}(d,p)$ with $\mathrm{SL}(d,p) \nleqslant H$ and that $|H| \geqslant p^{3d+1}$. Then there exists a d-generator p-group P of

- *exponent p,*
- *class at most 4,*
- *order at most $p^{\frac{d^4}{2}}$*

and such that $A(P) = H$.

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For $\mathcal{C}_9$, Saul Freedman: There is a class two group for $G_2(p) \leqslant GL(7, p)$, of order p^{14}.

Thanks!