Saturated fusion systems over p-groups with extraspecial subgroups of index p

Raul Moragues Moncho

University of Birmingham

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Theorem (Burnside, 1897)

Let G be a finite group with an abelian Sylow p-subgroup T. Then if $g, h \in T$ are fused in G they are also fused in $N_G(T)$.
We say $N_G(T)$ controls fusion in T.

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- Maps:

$$\mathrm{Hom}_{\mathcal{F}_T(G)}(P, Q) = \mathrm{Hom}_G(P, Q) = \{c_g \mid g \in G, P^g \leq Q\}$$

That is, the set of all maps $P \to Q$ induced by conjugation by elements of G. We denote $\mathrm{Aut}_{\mathcal{F}_T(G)}(P) = \mathrm{Hom}_{\mathcal{F}_T(G)}(P, P)$.

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We can generalise to the notion of saturated fusion system $\mathcal{F}$ on a p-group.

S_9

In S_9 we have subgroups of shape $S_3 \wr S_3 \cong C_3^3 \rtimes (S_4 \times C_2)$ (imprimitive in O'Nan-Scott), in which we can see that a Sylow 3-subgroup T is isomorphic to $C_3 \wr C_3$.
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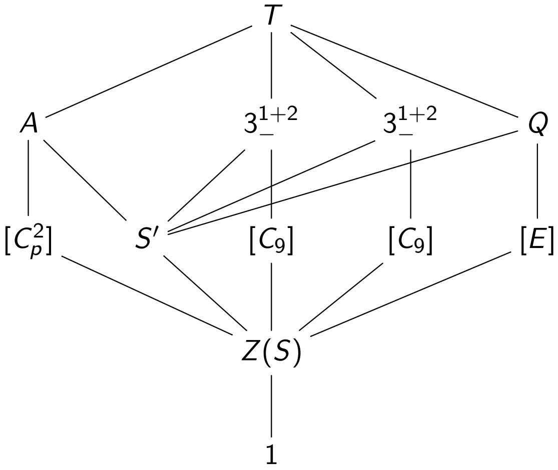
Can compare with $PSp_4(3)$ where we have isomorphic Sylow 3-subgroups and the parabolic subgroups of shape $C_3^3 \rtimes S_4$ and $3_+^{1+2} \rtimes SL_2(3)$.


T
A
3^{1+2}_-
3^{1+2}_-
Q
$[C_p^2]$
S'
$[C_9]$
$[C_9]$
$[E]$
$Z(S)$
1

Alperin's theorem

Theorem (Alperin-Goldschmidt fusion theorem, Puig)

Suppose $\mathcal{F}$ is a saturated fusion system on a p-group S. Then $\mathcal{F} = \langle Aut_{\mathcal{F}}(S), Aut_{\mathcal{F}}(E) \mid E \text{ is essential in } \mathcal{F} \rangle$.

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Given a fusion system $\mathcal{F}$ over a p-group S, a proper subgroup $E < S$ is *essential* in $\mathcal{F}$ if for every subgroup P that is $\mathcal{F}$-conjugate to E we have:

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2. $|N_S(E)| \geq |N_S(P)|$. We say E is fully $\mathcal{F}$-normalised;
3. $\mathrm{Out}_{\mathcal{F}}(E) = \mathrm{Aut}_{\mathcal{F}}(E)/\mathrm{Inn}(E)$ has a strongly p-embedded subgroup. That is there exists some $H < \mathrm{Out}_{\mathcal{F}}(E)$ such that for any nontrivial p-group $R \leq H$ we have $N_{\mathrm{Out}_{\mathcal{F}}(E)}(R) \leq H$.

Properties of essential subgroups

Lemma

Let E be essential in $\mathcal{F}$. Then

- *E is not cyclic.*
- *If E is abelian it is maximal abelian.*
- *$Out_{\mathcal{F}}(E)$ acts faithfully on $E/\Phi(E)$. Thus if E has rank r, then $Out_{\mathcal{F}}(E) \leq GL_r(p)$ and Sambale proved $|N_S(E)/E| \leq p^{\lfloor r/2 \rfloor}$.*

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For odd p there are examples by Ruiz-Viruel over 7_+^{1+2}, Oliver and Craven-Oliver-Semeraro when T has an (elementary) abelian subgroup of index p, and Parker-Stroth for S of order p^{p-1} with an extraspecial subgroup of index p.
From now on p is odd.

S with an extraspecial subgroup of index p (p odd)

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A p-group Q is *extraspecial* if $Z(Q) = \Phi(Q) = Q'$ is cyclic of order p. We denote it by $p_{\pm}^{1+2k}$.

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- *If $|S| = p^6$ then S is isomorphic to a Sylow p-subgroup of $PSL_4(p)$, $PSU_4(p)$ or $G_2(p)$. For $p \geq 11$ these are the only groups.*

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- *If $|S| = p^4$ then either S is isomorphic to a Sylow p-subgroup of $PSp_4(p)$ or $p = 3$.*

Results

Definition

$P \trianglelefteq \mathcal{F}$ if $P\phi = P$ for every $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ and $P \leq E$ for all essentials E. $O_p(\mathcal{F})$ is the largest normal subgroup of $\mathcal{F}$.

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Lemma

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Further assume that $|S| \geq p^6$. Then if $E < Q$, E is not essential.

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Proposition

Suppose $|S| = p^4$. The simple fusion systems on S are known, and there are exotic families. Classified by Oliver and Craven-Oliver-Semeraro.

Theorem

Suppose S is a p-group with $|S| \geq p^6$ containing an extraspecial subgroup Q of index p, and $\mathcal{F}$ a saturated fusion system on S with $O_p(\mathcal{F}) = 1$. Then either $|S| = p^6$ or $|S| = p^{p-1}$.

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  - *$PSL_4(p)$, classification in progress.*
- *If $|S| = p^{p-1}$ then S has maximal class and there is an essential subgroup of order p^2, a pearl.*

Future Work

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- When $|S| = p^{p-1}$ use the theory of pearls by Grazian.
- Prove uniqueness of the fusion systems.

Thank you