Cayley–Abels graphs for totally disconnected locally compact groups

Rögnvaldur G. Möller
University of Iceland

Groups St Andrews in Birmingham,
August 7, 2017

HÁSKÓLI ÍSLANDS
RAUNVÍSINDADEILD

Cayley–Abels graph

We start with a totally disconnected locally compact group (i.e. a tdlc group) G.

Definition

If G acts transitively on a connected locally finite connected graph Γ such that stabilizers of vertices are compact open subgroups then we say that Γ is a Cayley–Abels graph *for G.*

(The group acts from the right so that the image of a vertex v in Γ under an element g in G is written as vg.)

Examples

The p-adic numbers and p-adic Lie groups are examples of totally disconnected locally compact groups.

The automorphism group of a locally finite connected graph, with the topology of pointwise convergence is also such a group

If the action is transitive on the vertex set and the graph is connected then the graph is a Cayley–Abels graph for the group.

Existence and construction

A Cayley–Abels graph exists for G if and only if G is compactly generated.

Construction I. Take a compact generating set S.
Form the ordinary Cayley graph $\Gamma(G, S)$.
(G acts by automorphisms from the right.)
Then take a compact open subgroup U.
Define Γ as the quotient graph $\Gamma(G, S)/U$.
(U acts from the left,
vertices in the quotient graph are right cosets of U.)
Then Γ is a Cayley–Abels graph for G.

Another construction

Construction II. Pick a compact open subgroup U.
Find group elements $s_1, \ldots, s_n$ such that $U \cup \{s_1, \ldots, s_n\}$ generates G.
The vertex set of our Cayley–Abels graph Γ is the set of cosets G/U.
The vertex $\alpha = U$ will be adjacent to the vertices

$$\beta_1 = Us_1, \ \ldots \ , \beta_n = Us_n.$$

The edge set of Γ is

$$\mathrm{E}\Gamma = \{\alpha, \beta_1\}G \cup \cdots \cup \{\alpha, \beta_n\}G.$$

How does it work?

Similar to ordinary Cayley graphs. Possible to do "Geometric Group Theory".

Any two Cayley–Abels graphs for G are quasi-isometric.

Can use the Cayley–Abels graph to define invariants of the group, like the number of ends and growth.

Get analogues of Stallings Ends Theorem, results about Polynomial growth etc.

Minimal valency of Cayley–Abels graphs

Let G be a compactly generated totally disconnected locally compact group, i.e. a cgtdlc group.

Questions of George Willis: Does the minimal possible valency of a Cayley–Abels graph for G say something about G?
What do various properties of the group say about the minimal valency?

This minimal value is of course an invariant of the group. The minimal valency of a Cayley graph for a finitely generated group is in essence just the minimal number of generators. This is more involved.

Minimal number of orbits on edges

Question: What does the minimal possible number of orbits of G on the edges of any Cayley–Abels graph say about G? What do various properties of the group say about the minimal number of orbits on edges?

Looking at the second construction of a Cayley–Abels graph then the minimal number of orbits on edges is the same as the minimal number n such that there exists a compact open subgroup U and group elements $s_1, \ldots, s_n$ such that

$$\langle U \cup \{s_1, \ldots, s_n\}\rangle = G.$$

Minimal valency and one edge orbit

Theorem. *Suppose Γ is a Cayley–Abels graph of minimal valency for a cgtdlc group G. Suppose that v is a vertex in Γ and the stabilizer G_v acts transitively on $N(v)$ (the set of vertices adjacent to v). Then the action of G_v on $N(v)$ is 2-homogeneous, i.e. transitive on 2-element subsets. If the valency of Γ is even then the action is 2-transitive.*

Groups with low minimal valency

The minimal valency is 0 if and only if G is compact.
Minimal valency 1 does not occur.

Theorem. *The following are equivalent for a compactly generated totally disconnected locally compact group:*
(i) *The minimal valency of a Cayley–Abels graph is 2 (the Cayley–Abels graph is then an infinite line);*
(ii) *G has precisely two ends;*
(iii) *there is a continuous homomorphism with compact open kernel from G onto the infinite cyclic group or the infinite dihedral group*
(iv) *G has a co-compact infinite cyclic discrete subgroup.*

Minimal valency equal to 3

Is there something special then?

The scale function

Let G be a totally disconnected locally compact group. The scale function $s : G \to \mathbb{N}$ is an important part of Willis's structure theory for totally disconnected locally compact group and is defined by the formula

$$s(g) = \min |U : U \cap g^{-1}Ug|$$

where the minimum is taken over all compact-open subgroups U.

The scale function

Let G be a totally disconnected locally compact group. The scale function $s : G \to \mathbb{N}$ is an important part of Willis's structure theory for totally disconnected locally compact group and is defined by the formula

$$s(g) = \min |U : U \cap g^{-1}Ug|$$

where the minimum is taken over all compact-open subgroups U.

Let U be an arbitrary compact open subgroup and $\Omega = G/U$. Choose $\alpha \in \Omega$ such that $G_\alpha = U$. Another formula for the scale function is

$$s(g) = \lim_{n\to\infty} |U : U \cap g^{-n}Ug^n|^{1/n} = \lim_{n\to\infty} |(\alpha g^n)G_\alpha|^{1/n}.$$

Minimal valency 3 and the scale function

Theorem

Let G be a non-discrete compactly generated totally disconnected locally compact group. Suppose the minimal valency of a Cayley–Abels graph is 3. Then the scale function of G is non-trivial (i.e. takes other values than 1). Furthermore, if $g \in G$ is such that $s(g) \neq 1$ then $s(g) = 2^{k_g}$ for some integer $k_g \geq 1$.

Minimal valency 3 and the scale function

Theorem

Let G be a non-discrete compactly generated totally disconnected locally compact group. Suppose the minimal valency of a Cayley–Abels graph is 3. Then the scale function of G is non-trivial (i.e. takes other values than 1). Furthermore, if $g \in G$ is such that $s(g) \neq 1$ then $s(g) = 2^{k_g}$ for some integer $k_g \geq 1$.

Remark. The study of group actions on graphs was started in earnest in Tutte's papers in 1947 and 1959 on cubic graphs. The above theorems are proved by adapting Tutte's arguments.

The modular function

Let G be a totally disconnected locally compact group and μ a right invariant Haar-measure. The *modular function* on G is a homomorphism from G to the multiplicative group of positive real numbers such that if $g \in G$ and A is a measurable subset in G then $\mu(gA) = \Delta(g)\mu(A)$.

Let U be a compact open subgroup and $\Omega = G/U$. Then

$$\Delta(g) = \frac{|(\alpha g)G_\alpha|}{|\alpha G_{\alpha g}|}$$

is a homomorphism.

The modular function

This formula was discovered independently by Schlichting and Trofimov. Praeger showed in a purely permutation group theoretic context that the function defined by the right hand side defines a formula.

The results below on the modular function are contained in the 2016 master thesis of Arnbjörg Soffía Árnadóttir at the University of Iceland.

More on the modular function

Suppose G acts arc- and vertex transitively with compact open stabilizers of vertices on a digraph Γ with finite in- and out-valencies. If α is a vertex and $(\alpha, \alpha g)$ is an arc then

$$\Delta(g) = \frac{\text{out-valency}}{\text{in-valency}}.$$

The modular function is also related to the scale function since

$$\Delta(g) = \frac{s(g)}{s(g^{-1})}.$$

Still more on the modular function

Using an observation of Bass and Kulkarni one sees that if G acts on a Cayely–Abels graph then the modular function can be read of from the graph. Think of each edge as consisting of two directed edges and then label the edge with the quotient of the out- and in-valencies of the relevant orbital graph. Then $\Delta(g)$ is the multiple of all the labels in a directed path from α to αg.

Minimal valency and the modular function

If G is a compactly generated totally disconnected locally compact group and the image of the modular function is $\langle p/q \rangle$ where p and q are co-prime then the minimal valency of a Cayley–Abels graph is at least $p + q$.

Let T_k denote the k regular tree. Consider the subgroup G_ω of $\mathrm{Aut}\ T_k$ that fixes a given end ω of the tree. If $g \in G_\omega$ and α and αg are adjacent then $\Delta(g) = k - 1$.
The subgroup G_ω of the automorphism group of a k-regular tree has minimal valency k.

Scale function on a compactly generated group

Willis proved in 2001:

Let G be a compactly generated totally disconnected locally compact group. Then there are only finitely many distinct primes that occur as prime factors of the values that the scale function takes.

Scale function and minimal valency

Willis's result can be proved by looking at a Cayley–Abels graph and using the second formula for the scale function. This method also leads to

Scale function and minimal valency

Willis's result can be proved by looking at a Cayley–Abels graph and using the second formula for the scale function. This method also leads to

If p is the largest prime factor that occurs as factor in the values of s, then the minimal valency of a Cayley–Abels graph is at least $p+1$.

From this we can deduce that if p is a prime and G is the automorphism group of the $p+1$-regular tree then the minimal valency of a Cayley–Abels graph for G is $p+1$. (The values of the scale function on G are all powers of p and the scale function takes the value p.)

Automorphism groups of trees

Let T_k denote the regular tree of valency k. Any vertex- and boundary-transitive subgroup of $\mathrm{Aut}\ T_k$ has T_k as a minimal valency Cayley-Abels graph.

Let now T_{k_1,k_2} denote the semi-regular tree with valencies k_1 and k_2. Suppose G is a closed subgroup of the automorphism group that acts with two orbits on the vertices and is transitive on the boundary. A minimal valency Cayley–Abels graph for G is the 1-connected graph such that each block is either isomorphic to K_{k_1} or K_{k_2} and each vertex is contained in precisely one block of each type. This is the line-graph of T_{k_1,k_2}.

Conclusions

The results presented here show that the minimal valency of a Cayley–Abels graph relates in various non-trivial ways to the properties of the group.

To me this was a pleasant surprise.

Thank you for your attention!