Mirror Automorphisms of Chiral Regular Maps

Adnan Melekoğlu

Department of Mathematics
Faculty of Arts and Sciences
Adnan Menderes University
Aydın, Turkey

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Regular maps

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$\mathrm{Aut}^{\pm}(\mathcal{M})$: Group of all automorphisms of $\mathcal{M}$

$\mathrm{Aut}^{+}(\mathcal{M})$: Group of orientation-preserving automorphisms of $\mathcal{M}$

These groups have presentations of the form

$$\langle P, Q, R \mid P^2 = Q^2 = R^2 = (PQ)^2 = (QR)^m = (RP)^n = \cdots = 1\rangle \tag{1}$$

and

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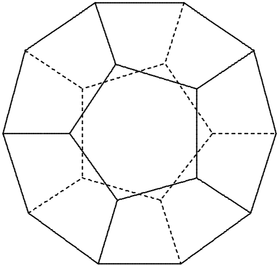
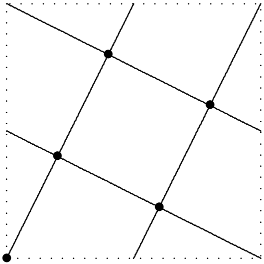
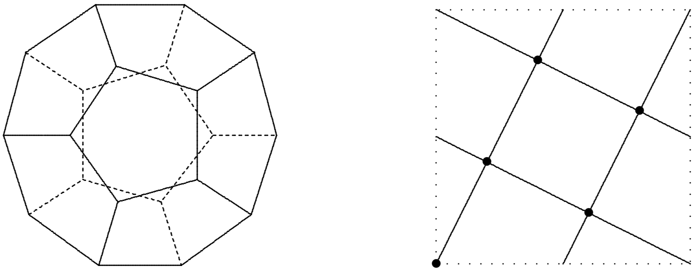
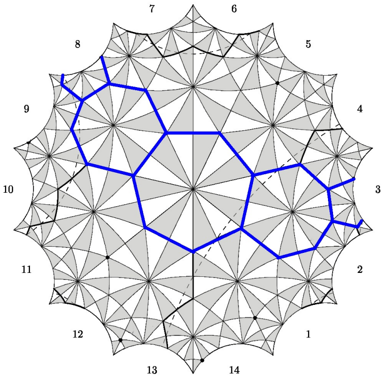
If $\mathrm{Aut}^+(\mathcal{M})$ is transitive on the directed edges, then $\mathcal{M}$ is called regular. If $\mathrm{Aut}^{\pm}(\mathcal{M})$ is transitive on the flags (incident vertex-edge-face triples), then $\mathcal{M}$ is called reflexible. If $\mathcal{M}$ is regular but not reflexible, then it is called chiral.







A reflexible regular map of type $\{5,3\}$ on the sphere and a chiral regular map of type $\{4,4\}$ on the square torus.



A reflexible regular map $\mathcal{M}$ of type $\{7, 3\}$ on Klein's surface of genus 3. (Edge pairings: 1–6, 2–11, 3–8, 4–13, 5–10, 7–12, 9–14.)
$\mathrm{Aut}^{+}(\mathcal{M}) \cong \mathrm{PSL}(2, 7)$

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Notation (Coxeter)

$\mathbf{0}$: vertex
$\mathbf{1}$: edge-centre
$\mathbf{2}$: face-centre

Patterns of Mirrors

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Every mirror M of a reflection of a regular map $\mathcal{M}$ passes through some of the geometric points of $\mathcal{M}$ such that these points form a periodic sequence, which is called the pattern of M. (By geometric points we mean the vertices, the face-centers and the edge-centers of $\mathcal{M}$.)

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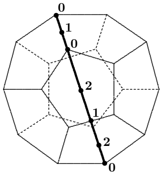
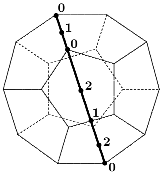
Example

Every mirror on the sphere fixed by a reflection of the map $\{5,3\}$ has pattern **010212010212** which we abbreviate to $(\mathbf{010212})^2$. So **010212** is a link and the link index is 2.

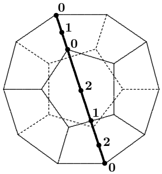

0
1
0
2
1
2
0



Theorem (Melekoğlu & Singerman, 2016)

(i) *The pattern of any mirror in a regular map* $\mathcal{M}$ *of type* $\{m, n\}$ *is obtained from one of the six links* **01**, **02**, **12**, **0102**, **0212** *and* **010212**;

(ii) *There cannot be more than three mirrors with different patterns on the same Riemann surface.*



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The possible patterns according to the parity of m and n are given in the following table.

Table : Patterns

Case	Pattern
m and n odd	$(\mathbf{010212})^{\ell}$
m odd n even	$(\mathbf{01})^{\ell_1}$
m odd n even	$(\mathbf{0212})^{\ell_2}$
m even n odd	$(\mathbf{12})^{\ell_1}$
m even n odd	$(\mathbf{0102})^{\ell_2}$
m and n even	$(\mathbf{01})^{\ell_1}$
m and n even	$(\mathbf{12})^{\ell_2}$
m and n even	$(\mathbf{02})^{\ell_3}$

Here ℓ, ℓ_1, ℓ_2 and ℓ_3 are the link indices and ℓ_is in different lines need not be equal.

How can we determine these patterns or link indices?

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Let X be a Riemann surface with underlying map $\mathcal{M}$ and let M be a mirror of a reflection of $\mathcal{M}$. A mirror automorphism of M is an automorphism of $\mathcal{M}$ that cyclically permutes the links of the pattern of M.

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Theorem (Melekoğlu & Singerman, 2016)

Each pattern corresponds to a conjugacy class of mirror automorphisms, and the order of the mirror automorphisms in that conjugacy class is equal to the link index of the pattern.

Table : Patterns and mirror automorphisms

Link	Mirror Automorphism
01	$C^{\frac{n}{2}}A$
02	$B^{\frac{m}{2}}C^{\frac{n}{2}}$
12	$B^{\frac{m}{2}}A$
0102	$C^{\frac{n+1}{2}}AC^{\frac{n-1}{2}}B^{\frac{m}{2}}$
0212	$C^{\frac{n}{2}}B^{\frac{m-1}{2}}AB^{\frac{m+1}{2}}$
010212	$B^{\frac{m-1}{2}}AB^{\frac{m+1}{2}}C^{\frac{n+1}{2}}AC^{\frac{n-1}{2}}$

Here A, B and C generate $\mathrm{Aut}^+(\mathcal{M})$ and satisfy the relations

$$A^2 = B^m = C^n = ABC = 1.$$

A mirror automorphism has no fixed-points, but if the genus is greater than one, then it fixes a curve setwise, which we call a fixed-curve.

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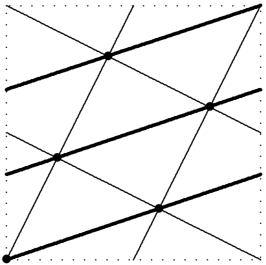

A fixed-curve with pattern $(\mathbf{02})^5$ on the square torus.

Number of Fixed-Curves

Theorem

Let $\mathcal{M}$ be a chiral regular map of type $\{m, n\}$ on a compact Riemann surface X and let $\|\mathcal{M}\|$ denote the number of fixed-curves on X. Then:

(i) *If m and n are odd, then* $\|\mathcal{M}\| = \frac{|\mathrm{Aut}^+(\mathcal{M})|}{2\ell}$;

(ii) *If m and n have different parities, then* $\|\mathcal{M}\| = \frac{|\mathrm{Aut}^+(\mathcal{M})|}{2}(\frac{1}{\ell_1} + \frac{1}{\ell_2})$;

(iii) *If m and n are even, then* $\|\mathcal{M}\| = \frac{|\mathrm{Aut}^+(\mathcal{M})|}{2}(\frac{1}{\ell_1} + \frac{1}{\ell_2} + \frac{1}{\ell_3})$.

Here ℓ, ℓ_1, ℓ_2 and ℓ_3 are the link indices and ℓ_is in different lines need not be equal.

Lengths of Fixed-Curves

Theorem

Let $\mathcal{M}$ be a chiral regular map of type $\{m, n\}$ on a compact Riemann surface X and let the lengths of the sides of a $(2, m, n)$-triangle be a, b and c as indicated in the figure below. Then the lengths of the fixed-curves can be determined by the formulae in the following table, where ℓ, ℓ_1, ℓ_2 and ℓ_3 are the link indices and ℓ_is in different lines need not be equal.

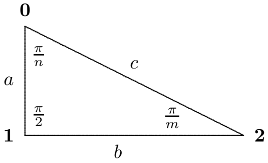

Table : Lengths of Fixed-curves

Case	Pattern	Length
m and n are odd	$(\mathbf{010212})^{\ell}$	$2\ell(a+b+c)$
m odd n even	$(\mathbf{01})^{\ell_1}$	$2\ell_1 a$
m odd n even	$(\mathbf{0212})^{\ell_2}$	$2\ell_2(b+c)$
m even n odd	$(\mathbf{12})^{\ell_1}$	$2\ell_1 b$
m even n odd	$(\mathbf{0102})^{\ell_2}$	$2\ell_2(a+c)$
m and n are even	$(\mathbf{01})^{\ell_1}$	$2\ell_1 a$
m and n are even	$(\mathbf{12})^{\ell_2}$	$2\ell_2 b$
m and n are even	$(\mathbf{02})^{\ell_3}$	$2\ell_3 c$

Thank You