New progress in products of conjugacy classes

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Joint work with Antonio Beltrán and María José Felipe

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Notation

Conjugacy class

Let G be a finite group and $x \in G$. We denote

$$x^G = \{x^g = g^{-1}xg : g \in G\}$$

the **conjugacy class** of x in G and $|x^G| = |G : \mathbf{C}_G(x)|$ is the class size of x. A conjugacy class K is **real** if $K = K^{-1} = \{x^{-1} : x \in K\}$. We denote by $\mathrm{cl}(G)$ the set of conjugacy classes of G.

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G-invariant set and product of two conjugacy classes

Let X be a G-invariant subset of G, that is $X^g = \{x^g | x \in X\} = X$ for all $g \in G$. Then X can be expressed as a union of conjugacy classes of G. If A and B are conjugacy classes, then $AB = \{ab : a \in A, b \in B\}$ is a G-invariant set.

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Product of two conjugacy classes

Problem

When is the product of two conjugacy classes a conjugacy class?

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Some elementary facts:

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- If $z \in \mathbf{Z}(G)$ and $A \in \mathrm{cl}(G)$, then $zA \in \mathrm{cl}(G)$.
- If $A, B \in \mathrm{cl}(G)$ and $(|A|, |B|) = 1$, then $AB \in \mathrm{cl}(G)$.

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- If $A, B \in \mathrm{cl}(G)$ and $(|A|, |B|) = 1$, then $AB \in \mathrm{cl}(G)$.
- If G is nilpotent and $x, y \in G$ with coprime orders, then $x^G y^G = (xy)^G \in \mathrm{cl}(G)$.

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Conjecture (Arad and Herzog, 1985)

If there exist $C, D \in \mathrm{cl}(G)$ such that $CD \in \mathrm{cl}(G)$, then G is not a non-abelian simple group.

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Theorem (Arad and Fisman, 1985)

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Product of two conjugacy classes

Simple groups satisfying Arad and Herzog's conjecture:

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Theorem (Moori, Tong-Viet, 2011)

The conjecture of Arad and Herzog is true for $\mathrm{PSL}(3,q)$, $\mathrm{PSU}(3,q)$, ${}^2\mathrm{G}_2(q)$, $\mathrm{PSp}(4,q)$.

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Lemma

Let G be a group and let a, $b \in G$ be non-trivial elements of G. The following conditions are equivalent:

- $a^G b^G = c^G$
- $\chi(a)\chi(b) = \chi(c)\chi(1)$ for all $\chi \in \mathrm{Irr}(G)$.

J. Moori and H.P. Tong-Viet, Products of conjugacy classes in simple groups. Quaest. Math., **34** (2011), 433-439.

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Square of a conjugacy class

Teorema (Guralnick-Navarro, 2016)

Let G be a finite group and $K = x^G$ the conjugacy class of x in G. The following assertions are equivalent:

1. K^2 is a conjugacy class of G.
2. $K = x[x, G]$ and $\mathbf{C}_G(x) = \mathbf{C}_G(x^2)$.
3. $\chi(x) = 0$ or $|\chi(x)| = \chi(1)$ for all $\chi \in \mathrm{Irr}(G)$, and $\mathbf{C}_G(x) = \mathbf{C}_G(x^2)$.

In this case, $[x, G]$ is solvable (CFSG is needed), and $\langle K \rangle = \langle x \rangle [x, G]$ too. Furthermore, if the order of x is a prime power, then $[x, G]$ has normal p-complement.

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Remark

If $K^2 = D \neq 1$, then K is non-real. What happens in the simplest real case $K^2 = 1 \cup D$?

Square of a conjugacy class

Theorem (B-F-M, 2017)

Let G be a finite group and $K = x^G$ the conjugacy class of x in G. Suppose that $K^2 = 1 \cup D$, where D is a conjugacy class of G. Then $\langle D \rangle = [x, G]$ is either cyclic or p-group for some prime p. Moreover, $\langle K \rangle = \langle x \rangle [x, G]$ is solvable (CFSG is not needed). More precisely,

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1. Suppose that $|K| = 2$.
    1. If $o(x) = 2$, then $\langle K \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_2 \cong \langle D \rangle \subseteq \mathbf{Z}(G)$.

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    2. If $o(x) = n > 2$, then $\langle K \rangle \cong \mathbb{Z}_n$ and $\langle D \rangle$ is cyclic.

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   2. If $o(x) = n > 2$, then $\langle K \rangle \cong \mathbb{Z}_n$ and $\langle D \rangle$ is cyclic.
2. Suppose that $|K| \geq 3$.
   1. If $o(x) = 2$, then either $\langle K \rangle$ and $\langle D \rangle$ are 2-elementary abelian groups or $\langle D \rangle$ is a p-group and $|K| = p^r$ with p an odd prime and $r \in \mathbb{Z}^+$.

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Let G be a finite group and $K = x^G$ the conjugacy class of x in G. Suppose that $K^2 = 1 \cup D$, where D is a conjugacy class of G. Then $\langle D \rangle = [x, G]$ is either cyclic or p-group for some prime p. Moreover, $\langle K \rangle = \langle x \rangle [x, G]$ is solvable (CFSG is not needed). More precisely,

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    1. If $o(x) = 2$, then either $\langle K \rangle$ and $\langle D \rangle$ are 2-elementary abelian groups or $\langle D \rangle$ is a p-group and $|K| = p^r$ with p an odd prime and $r \in \mathbb{Z}^+$.
    2. If $o(x) > 2$, then $\langle D \rangle$ is a p-elementary abelian group for some odd prime p. Furthermore, either $o(x) = p$ or $o(x) = 2p$.

A. Beltrán, M.J. Felipe and C. Melchor, Squares of real conjugacy classes in finite groups. Ann. Mat. Pura Appl. DOI 10.1007/s10231-017-0681-0

Square of a conjugacy class

Corollary (B-F-M, 2017)

Let K be a conjugacy class of a finite group G such that K^2 is union of conjugacy classes, all of which are central except at most one. Then $\langle K \rangle$ is solvable (CFSG is needed).

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Corollary (B-F-M, 2017)

Let G be a finite group such that every non-central conjugacy class K satisfies that $K^2 = 1 \cup D$, where D is a conjugacy class of G. Then $G/\mathbf{F}(G)$ is an elementary abelian 2-group.

Square of a conjugacy class

Corollary (B-F-M, 2017)

Let G be a finite group such that every conjugacy class K satisfies that K^2 is union of conjugacy classes, all of which central except at most one. Let $M/\mathbf{F}(G) = \mathbf{O}_2(G/\mathbf{F}(G))$. Then G/M is nilpotent and, consequently, G is solvable with Fitting length at most 3.

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Theorem (Chillag and Mann, 1998)

Let G be a finite group and suppose that K^2 is a conjugacy class for every conjugacy class K of G. Then G is nilpotent.

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Theorem (Chillag and Mann, 1998)

Let G be a finite group and suppose that K^2 is a conjugacy class for every conjugacy class K of G. Then G is nilpotent.

Corollary (B-F-M, 2017)

Let G be a finite group and let π be a set of primes. Suppose that K^2 is a conjugacy class for every conjugacy class K of π-elements of G. Then $G/\mathbf{O}_{\pi'}(G)$ is nilpotent. In particular, if $\pi = \pi(G)$, then G is nilpotent.

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Multiplying a conjugacy class by its inverse in a finite group

If $KK^{-1} = 1 \cup D \cup D'$, are $\langle K \rangle$ and $\langle KK^{-1} \rangle$ then solvable?

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Example

The answer is no. Actually, $\langle KK^{-1} \rangle$ may be even simple. For instance, if $G = S_n$ for any $n \geq 5$ and $K = (1\ 2)^{S_n}$, then

$$KK^{-1} = 1 \cup (1\ 2)(3\ 4)^{S_n} \cup (1\ 2\ 3)^{S_n}$$

and $\langle KK^{-1} \rangle = A_n$.

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Theorem, (B-F-M, 2017)

Let K be a non-trivial conjugacy class of a finite group G and suppose that $KK^{-1} = 1 \cup D \cup D^{-1}$, where D is a conjugacy class of G. Then G is not a non-abelian simple group. In particular, it holds if $KK^{-1} = 1 \cup D$.

Multiplying a conjugacy class by its inverse in a finite group

Lemma, (B-F-M, 2017)

Let G be a group and $x, d \in G$. Let $K = x^G$ and $D = d^G$. The following are equivalent:

1. $KK^{-1} = 1 \cup D \cup D^{-1}$
2. For every $\chi \in \mathrm{Irr}(G)$

$$|K||\chi(x)|^2 = \chi(1)^2 + \frac{(|K|-1)}{2}\chi(1)(\chi(d) + \chi(d^{-1})).$$

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If K is real, then $K^2 = 1 \cup D$ and the equivalent equation with characters is

$$|K|\chi(x)^2 = \chi(1)^2 + (|K|-1)\chi(1)\chi(d)$$

for every $\chi \in \mathrm{Irr}(G)$.

Multiplying a conjugacy class by its inverse in a finite group

$$\mathbf{K K^{-1} = 1 \cup D \cup D^{-1} \Rightarrow G \text{ is not simple}}$$

PARTS OF THE PROOF OF THE THEOREM:

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PARTS OF THE PROOF OF THE THEOREM:

- If $G \cong A_n$ with $n > 5$, we use the following theorem.

Theorem, (Adan-Bante, 2008)

Let S_n be the symmetric group of n-letters, $n > 5$ and α, $\beta \in S_n \setminus 1$. Then $\eta(\alpha^{S_n}\beta^{S_n}) \geq 2$ and, if $\eta(\alpha^{S_n}\beta^{S_n}) = 2$ then either α or β is a fixed point free permutation. Assume that α is fixed point free. Then one of the following holds:

- If $n = 2k$, α is the product of $n/2$ disjoint transpositions and β is either a transposition or a 3-cycle.
- If $n = 3k$, α is the product of $n/3$ disjoint 3-cycles and β is a transposition.

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Lemma, (Adan-Bante, 2009)

Let $\alpha, \beta \in A_n$ with $n \geq 6$. Assume that $\alpha^{A_n} \neq \alpha^{S_n}$ and $\beta^{A_n} \neq \beta^{Sn}$. Then $\eta(\alpha^{A_n}\beta^{A_n}) \geq 5$.

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PARTS OF THE PROOF OF THE THEOREM:

- If G is a sporadic simple group, we find some character that does not satisfy the characterisation by characters.
- If G is simple of Lie type over a field of characteristic p, we take the Steinberg character ψ, which satisfies:
  - $\psi(1) = |G|_p$.
  - If x is p-regular, then $\psi(x) = |\mathbf{C}_G(x)|_p$.
  - If x is p-singular, then $\psi(x) = 0$.

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Multiplying a conjugacy class by its inverse in a finite group

OPEN QUESTION: If $KK^{-1} = 1 \cup D \cup D^{-1}$, are $\langle D \rangle$ and $\langle K \rangle$ then solvable?

Multiplying a conjugacy class by its inverse in a finite group

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Theorem, (B-F-M, 2017)

Let K a non-real conjugacy class of G satisfying $KK^{-1} = 1 \cup K \cup K^{-1}$. Then $\langle K \rangle$ is an elementary abelian p-group with p an odd prime such that if $|\langle K \rangle| = p^n$, then $p \equiv 3 \ mod \ 4$ and n is odd. Moreover, $|K| = \frac{p^n-1}{2}$.

Multiplying a conjugacy class by its inverse in a finite group

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If S is a G-invariant set, then we can write

$$\widehat{S} = \sum_{g \in S} g = \sum_{i=1}^{n} n_i \widehat{K_i}$$

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We denote by $n_i = (\widehat{S}, \widehat{K_i}) = (\widehat{K_i}, \widehat{S})$ the multiplicity of $\widehat{K_i}$ in $\widehat{S}$.

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Properties

If D_1, D_2 and D_3 are conjugacy classes of a finite group G, then

1. $(\widehat{D_1}\widehat{D_2}, \widehat{D_3}) = (\widehat{D_1^{-1}}\widehat{D_2^{-1}}, \widehat{D_3^{-1}})$
2. $(\widehat{D_1}\widehat{D_2}, \widehat{D_3}) = |D_2||D_3|^{-1}(\widehat{D_1}\widehat{D_3^{-1}}, \widehat{D_2^{-1}})$
3. $(\widehat{D_1}\widehat{D_2}, \widehat{D_1}) = |D_2||D_1|^{-1}(\widehat{D_1}\widehat{D_1^{-1}}, \widehat{D_2^{-1}}) = (\widehat{D_2}\widehat{D_1^{-1}}, \widehat{D_1^{-1}}) = (\widehat{D_2^{-1}}\widehat{D_1}, \widehat{D_1})$.

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Theorem, (B-F-M, 2017)

Let K be a conjugacy class of a finite group G and suppose that $KK^{-1} = 1 \cup D$, where D is a conjugacy class of G. Then $|D|$ divides $|K|(|K| - 1)$ and $\langle K \rangle / \langle D \rangle$ is cyclic and

1. If $|D| = |K| - 1$, then $\langle K \rangle$ is metabelian. More precisely, $\langle D \rangle$ is p-elementary abelian for some prime p.
2. If $|D| = |K|$, then $\langle K \rangle$ is solvable with derived length at most 3.
3. If $|D| = |K|(|K| - 1)$, then $\langle K \rangle$ is abelian.

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Thank you for your attention!

New progress in products of conjugacy classes

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