Code algebras, axial algebras and VOAs

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Motivation

- Vertex operator algebras (VOAs)
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    - Mathematicians noticed some intriguing links between finite groups and modular functions, two apparently unrelated mathematical objects dubbed Monstrous Moonshine. This led to the moonshine VOA $V^\natural$.
    - Code VOAs are an important class where a binary linear code governs the representation theory of the VOA.
    - All framed VOAs V (such as $V^\natural$) have a unique code sub VOA and V is a simple current extension of its code sub VOA.

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$$A = A_1 \oplus A_0 \oplus A_{\lambda_1} \oplus \cdots \oplus A_{\lambda_k}$$

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$$A = A_1 \oplus A_0 \oplus A_{\lambda_1} \oplus \cdots \oplus A_{\lambda_k}$$

    where A_λ is the λ-eigenspace for ad_a.
  - All the idempotents in the given generating set satisfy the same set of *fusion rules* which are a table of where the product of an element of A_λ with an element of A_μ lies.

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- We get interesting non-associative algebras!

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The *code algebra* $A = A_C(a, b, c)$ is the free commutative algebra over $\mathbb{F}$ on the basis

$$\{t_i : i = 1, \ldots, n\} \cup \{e^\alpha : \alpha \in C^*\},$$

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$$\{t_i : i = 1, \dots, n\} \cup \{e^\alpha : \alpha \in C^*\},$$

where $C^* := C - \{0, 1\}$, modulo the relations

$$t_i \cdot t_j = \delta_{i,j}$$

$$t_i \cdot e^\alpha = \begin{cases} a\, e^\alpha & \text{if } \alpha_i = 1 \\ 0 & \text{if } \alpha_i = 0 \end{cases}$$

$$e^\alpha \cdot e^\beta = \begin{cases} b\, e^{\alpha+\beta} & \text{if } \alpha \neq \beta, \beta^c \\ c \displaystyle\sum_{i \in \mathrm{supp}(\alpha)} t_i & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha = \beta^c \end{cases}$$

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Let A_C be a non-degenerate code algebra.

- *If $C = \{0, 1, \alpha, 1+\alpha\}$, then A_C has exactly two non-trivial proper ideals,*
- *otherwise A_C is simple.*

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Code algebras have a large automorphism group which contains a group of the form $M{:}\mathrm{Aut}(C)$, where M is generated by involutions coming from some idempotents.

Frobenius form

Definition

A Frobenius form on a code algebra is a symmetric bilinear form $(\cdot, \cdot) : A \times A \to \mathbb{F}$ such that

1. *the form associates. That is, $(x, yz) = (xy, z)$ for all $x, y, z \in A$.*
2. *$(a, a) = (b, b)$ for all idempotents a and b with the same fusion rules.*

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2. $(a, a) = (b, b)$ *for all idempotents* a *and* b *with the same fusion rules.*

Theorem

Let A *be a non-degenerate code algebra. Then* A *admits a unique Frobenius form (up to scaling) and it is given by:*

$$(t_i, t_j) = \delta_{i,j}$$
$$(t_i, e^\alpha) = 0$$
$$(e^\alpha, e^\beta) = \frac{c}{a}\delta_{\alpha,\beta}$$

Idempotents

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	1	0	a
1	1		a
0		0	a
a	a	a	$1, 0$

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Lemma (s-map construction)

There exists an idempotent

$$s(D, v) := \lambda \sum_{i \in \mathrm{supp}(D)} t_i + \mu \sum_{\alpha \in D^*} (-1)^{(v,\alpha)} e^\alpha$$

where $\lambda, \mu \in \mathbb{F}$ satisfy a linear and quadratic equation, respectively.

Small idempotents

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$$e_{\pm} := \lambda \sum_{i \in \mathrm{supp}(\alpha)} t_i \pm \mu e^{\alpha}$$

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Corollary

A non-degenerate code algebra is generated by idempotents if $ac > \frac{c}{2|\alpha|}$.

Small idempotents

Theorem

If $ac > \frac{c}{2|\alpha|}$ and $a \neq \frac{1}{3|\alpha|}$ then the small idempotents exist, are primitive and semi-simple, and have fusion rules given by

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	1	0	λ	$\frac{2\lambda-1}{2}$	ν_+	ν_-
1	1		λ	$\frac{2\lambda-1}{2}$	ν_+	ν_-
0		0			ν_+	ν_-
λ	λ		$1, \lambda, \frac{2\lambda-1}{2}$		ν_-	ν_+
$\frac{2\lambda-1}{2}$	$\frac{2\lambda-1}{2}$			$1, \frac{2\lambda-1}{2}$	ν_+	ν_-
ν_+	ν_+	ν_+	ν_-	ν_+	$1, 0, \lambda, \frac{2\lambda-1}{2}, \nu_+$	$0, \lambda$
ν_-	ν_-	ν_-	ν_+	ν_-	$0, \lambda$	$1, 0, \lambda, \frac{2\lambda-1}{2}, \nu_-$

where $\nu_\pm := \frac{1}{4} \pm \mu b$.

Axial algebras

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Corollary

Let C be a simplex or first order Reed-Muller code and $A_C(a, b, c)$ be a non-degenerate code algebra with $ac > \frac{c}{2|\alpha|}$ and $a \neq \frac{1}{3|\alpha|}$. Then, A is an axial algebra.

Hamming code example

Let $C = H_8$ be the extended Hamming code and $(a, b, c) = (\frac{1}{4}, \frac{1}{2}, 1)$.

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The s-map construction gives two additional sets of eight mutually orthogonal idempotents, $s(C, v)$, one for v with odd weight, one for v even weight.

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Moreover, these have the same fusion rules as the t_i making A_{H_8} an axial algebra (of Jordan type). And it has automorphism group containing

$$2^6{:}(PSL_3(2) \times S_3).$$

Thank you for listening!

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