Homomorphisms between restricted genera

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Homomorphisms

Constructing a map that preserves algebraic structure is a natural exercise when dealing with sets having interesting algebraic structure and presents computations advantages.
Aim of the study

We focus on the class $\mathcal{X}_0$ of all finitely generated groups with finite commutator subgroup. Given two such groups $G_1$ and $G_2$ for which $n_1$ and $n_2$ are relatively prime, we aim at establishing a homomorphism between localization genera of such groups under a given finite group $F$. 
Introduction

Localization Theory

The theory of $\pi$-localization of groups, where $\pi$ is a family of primes, appears to have been first discussed in [10, 9] by Mal’cev and Lazard and many others become interested in the theory, such as Baumslag [1, 2] and Bousfield-Kan [3]

Genus of a group

In the 1970s, Hilton and Mislin became interested through their work on the localization of nilpotent spaces, in the localization of nilpotent groups.


Definition

Mislin in [12] defines the genus of a finitely generated nilpotent group $G$ denoted by $\mathcal{G}(G)$, to be the set of all isomorphism classes of finitely generated nilpotent groups $H$ such that $G_p \cong H_p$ for every prime number $p$.

Hilton and Mislin in [7] defined an abelian group structure on the genus set $\mathcal{G}(G)$ of a finitely generated nilpotent group $G$ with finite commutator subgroup.

Definition

For a finitely generated group $G$ with finite commutator subgroup, the **non-cancellation set** is the set $\chi(G)$ of all isomorphism classes of finitely generated group $H$ such that $G \times \mathbb{Z} \cong H \times \mathbb{Z}$.

The set $\tau_f(G)$ of all isomorphism classes of finitely generated group $H$ such that $G_{\pi} \cong H_{\pi}$ for every finite set of primes $\pi$ is called the **restricted genus** of $G$. 
Assigning a natural number $n(G)$ to a $X_0$-group $G$

Let $n_1$ be the exponent of $T_G$, let $n_2$ be the exponent of the group $\text{Aut}(T_G)$, and let $n_3$ be the exponent of the torsion subgroup of the centre of $G$. Consider $n(G) = n_1n_2n_3$.

$n = n(G)$ has the property that the subgroup $G^{(n)} = \langle g^n : g \in G \rangle$ of $G$ belongs to the centre of $G$ and $G/G^{(n)}$ is a finite group.
Let $\pi = \{ p : p \text{ is a prime and } p | n(G) \}$. Then the short exact sequence

$$1 \to G^{(n)} \to G \to G/G^{(n)} \to 1$$

determines $G$ as an extension of a $\pi'$-torsion-free finitely generated abelian group $G^{(n)}$ by a $\pi$-torsion group $G/G^{(n)}$. From [?, Proposition 3.1], it follows that the $\pi$-localization homomorphism

$$G \to G_\pi$$

is injective.
Witbooi in [16] shows that the non-cancellation set of a $\mathcal{X}_0$-group $G$ has a group structure and there is an epimorphism

$$\zeta : \mathbb{Z}_n^* / \pm 1 \rightarrow \chi(G),$$

For a nilpotent $\mathcal{X}_0$-group $G$, Warfield in [15] shows that

$$\chi(G) \cong \mathcal{G}(G)$$

O'Sullivan in [13] shows that for a $\mathcal{X}_0$-group $G$,

$$\chi(G) \cong \tau_f(G)$$


Existence of homomorphisms

For a semidirect product $H = \mathbb{Z}_m \rtimes \omega \mathbb{Z}$, the authors in [5] showed that there is a well-defined surjective homomorphism

$$\Gamma : \chi(H) \to \chi(H^r)$$

given by $[K] \to [K \times H^{r-1}]$ where $K$ is a group such that $K \times \mathbb{Z} \cong H \times \mathbb{Z}$ and $r$ is a natural number. Thus, in order to compute the group $\chi(H^r)$ one needs only to compute the kernel of the homomorphism $\Gamma$.

Computation of $\chi(G_1 \times G_2)$


**Description**

Witbooi in [16] notice that for any $\mathcal{X}_0$-groups $G_1$ and $G_2$ and for groups $K$ belonging to $\chi(G_1)$, the rule $K \mapsto K \times G_2$ induces a well-defined function $\theta : \chi(G_1) \rightarrow \chi(G_1 \times G_2)$ which is an epimorphism.
Let us fix a finite group $F$. Let $\text{Grp}_F$ be the category of groups under $F$. Here we mean that the objects of $\text{Grp}_F$ are group homomorphisms $\varphi : F \to G$.

Given another object $\varphi_1 : F \to G_1$, a morphism in $\text{Grp}_F$ corresponds to a group homomorphism $\alpha : G \to G_1$ such that $\alpha \circ \varphi = \varphi_1$.

For a set of primes $\pi$, the $\pi$-localization of an object $\varphi : F \to G$ will be the object $\varphi_\pi : F \to G_\pi$. Then localization is an endofunctor of $\text{Grp}_F$. 
Let $\mathcal{X}_F$ be the full subcategory of $\mathcal{X}_0$-groups under $F$. We can define the restricted genus
$$\Gamma_f(\varphi) = \{[\psi] \mid \psi_\pi \text{ is isomorphic to } \varphi_\pi\}$$
If $F$ is the trivial group, then $\mathcal{X}_F$ can be identified with the class $\mathcal{X}_0$ of groups.
In line with [16] and in analogy with $\mathcal{X}_0$-groups we shall write
$$\Gamma_f(\phi) = \chi(G, \phi).$$
Theorem [11, Theorem 2.3]

Let \((L, l)\) be an object representing a member of \(\chi(G, h)\). Then there exist a subgroup \(J\) of \(G\) with \([G : J]\) finite and \([G : J]\) relatively prime to \(n\), such that in \(\text{Grp}_F\) the object \(F \rightarrow J\) is isomorphic to \((L, l)\).

Homomorphisms between non-cancellation groups

Existence of homomorphisms

Let $F$ be a finite group and consider the homomorphism $h : F \rightarrow G$. In [11], a group structure is defined on $\chi(G, h)$ and an epimorphism

$$\zeta : (\mathbb{Z}/n)^* / \pm 1 \rightarrow \chi(G, h)$$

is established.

It is also shown that there exist natural epimorphisms

$$\chi(G, h) \rightarrow \chi(G/h(F))$$

and

$$\chi(G, h) \rightarrow \chi(G, h \circ i))$$

Non-existence of homomorphisms

In [11], computation methods of $\chi(G, h)$ in the special case $G$ is a semidirect product $T \rtimes \omega \mathbb{Z}^k$ are used in a very particular example to provide a concrete computation of $\chi(G, h)$. It is used to show that there doesn’t exist any homomorphism $\gamma$ to make the following diagram commutative

\[
\begin{array}{ccc}
\chi(K, h) & \xrightarrow{\alpha} & \chi(K/h(F)) \\
\beta & \downarrow & \gamma \\
\chi(K) & & \\
\end{array}
\]

Fix any $m \in \mathbb{N}$. Let $X(m) = \{u \in \mathbb{N} | (u, m) = 1\}$. Now consider any $G \in X_0$ and let $n = n(G)$. Let $Y(G, h)$ be the set of all $u \in X(n)$ for which there exists a subgroup $J$ of $G$ with $[G : J] = u$ and such that the object $(J, h_J)$ represents a member of $\chi(G, h)$. Here $h_J$ is the induced homomorphism obtained from $h$ by restriction of the codomain. For each $u \in Y(G, h)$, let us choose a subgroup $G_u$ of $G$ such that $T_G \subseteq G_u$ and $[G : G_u] = u$. Let $h_u : F \rightarrow G_u$ be the induced homomorphism defined by $h_u : x \mapsto h(x)$. Now let us denote the isomorphism class of the object $h_u$ of $X_F$ by $[G_u, h_u]$. Then we obtain a function $\xi : Y(G, h) \rightarrow \chi(G, h)$. Let $Y^*(G, h)$ denote the image of $Y(G, h)$ in $\mathbb{Z}_n^*$. 
Theorem [11, Theorem 2.5]

\begin{itemize}
  \item [a)] $Y^*(G, h)$ is a subgroup of $\mathbb{Z}_n^*$.
  \item [b)] The function $\xi$ induces a (well-defined) function

  \[ \zeta : Y^*(G, h)/\pm 1 \to \chi(G, h). \]

  \item [c)] The fibre $\zeta^{-1}[G, h]$ of $\zeta$ over $[G, h]$ is a subgroup of $Y^*(G, h)/\pm 1$.
  \item [d)] For any $[K, k] \in \chi(G, h)$, $\zeta^{-1}[K, k]$ is a coset of $\zeta^{-1}[G, h]$.
\end{itemize}

Theorem

Let \((G_1, h_1)\) and \((G_2, h_2)\) be such that \(n_1 = n(G_1)\) and \(n_2 = n(G_2)\) are relatively prime. Then,

1. There is a homomorphism

   \[ \alpha : Y^*(G_1, h_1)/\pm 1 \to Y^*(G_2, h_2)/\pm 1 \]

   defined by \(u \mapsto n_1 \lceil \ln(u) \rceil \).

2. There are homomorphisms \(\varphi\) and \(\beta\) such that the following diagram is commutative.

\[
\begin{array}{ccc}
Y^*(G_1, h_1)/\pm 1 & \xrightarrow{\xi_1} & \chi(G_1, h_1) \\
\downarrow \beta & & \downarrow \varphi \\
\chi(G_2, h_2) & & 
\end{array}
\]
Corollary

Let \((G_1, h_1)\) and \((G_2, h_2)\) be such that \(n_1 = n(G_1)\) and \(n_2 = n(G_2)\) are relatively prime. Let \(\varphi\) and \(\beta\) the following homomorphisms

\[
Y^*(G_1, h_1)/ \pm 1 \xrightarrow{\xi_1} \chi(G_1, h_1) \xrightarrow{\varphi} \chi(G_2, h_2)
\]

If \(\beta\) is surjective, then \(\varphi\) is an epimorphism.
Proposition [16, Proposition 6.1] Suppose that we have groups $A$, $B$ and $C$ together with a homomorphism $\beta : A \to C$ and a surjective group homomorphism $\gamma : A \to B$. If $\alpha : B \to C$ is a function (between sets) such that $\alpha \circ \gamma = \beta$, then $\alpha$ is a homomorphism. Moreover, if $\beta$ is surjective, then $\alpha$ is also surjective.

THANK YOU


