Homomorphisms between restricted genera

Jules C. Mba

Department of Pure and Applied Mathematics
University of Johannesburg
South Africa

jmba@uj.ac.za

August 11, 2017

Introduction

Homomorphisms

Constructing a map that preserves algebraic structure is a natural exercise when dealing with sets having interesting algebraic structure and presents computations advantages.

Introduction

Aim of the study

We focus on the class $\mathcal{X}_0$ of all finitely generated groups with finite commutator subgroup. Given two such groups G_1 and G_2 for which n_1 and n_2 are relatively prime, we aim at establishing a homomorphism between localization genera of such groups under a given finite group F.

Introduction

Localization Theory

The theory of π-localization of groups, where π is a family of primes, appears to have been first discussed in [10, 9] by Mal'cev and Lazard and many others become interested in the theory, such as Baumslag [1, 2] and Bousfield-Kan [3]

[9]M. Lazard, Sur les groupes nilpotents et les anneaux de Lie, *(French) Ann. Sci. Ecole Norm. Sup.* (3) **71** (1954) 101-190.

[10] A.I. Mal'cev, Nilpotent torsion-free groups, *(Russian) Izvestiya Akad. Nauk. SSSR. Ser. Mat.* **13** (1949) 201-212.

[1]G. Baumslag, Lecture notes on nilpotent groups, *Regional Conference Series in Mathematics*, No. 2, *American Mathematical Society, Providence, R.I.* 1971.

[2]G. Baumslag, Some remarks on nilpotent groups with roots, *Proc. Amer. Math. Soc.* **12** (1961) 262-267.

[3]A.K. Bousfield and D. M. Kan, Homotopy limits, completions and localizations, *Lecture Notes in Mathematics*, Vol. 304. Springer-Verlag, Berlin-New York, 1972.

Genus of a group

In the 1970s, Hilton and Mislin became interested through their work on the localization of nilpotent spaces, in the localization of nilpotent groups.

[12] G. Mislin, Nilpotent groups with finite commutator subgroups, Localization in group theory and homotopy theory, and related topics (*Sympos., Battelle Seattle Res. Center, Seattle, Wash.*, 1974), 103-120, Lecture Notes in Math., Vol. 418, *Springer, Berlin*, 1974.

Definition

Mislin in [12] defines the **genus** of a finitely generated nilpotent group G denoted by $\mathcal{G}(G)$, to be the set of all isomorphism classes of finitely generated nilpotent groups H such that $G_p \cong H_p$ for every prime number p.

Hilton and Mislin in [7] defined an abelian group structure on the genus set $\mathcal{G}(G)$ of a finitely generated nilpotent group G with finite commutator subgroup.

[7] P. Hilton and G. Mislin, On the genus of a nilpotent group with finite commutator subgroup, *Math. Z.* **146** (1976), no. 3, 201-211.

Non-cancellation set of a group

Definition

For a finitely generated group G with finite commutator subgroup, the **non-cancellation set** is the set $\chi(G)$ of all isomorphism classes of finitely generated group H such that $G \times \mathbb{Z} \cong H \times \mathbb{Z}$.

The set $\tau_f(G)$ of all isomorphism classes of finitely generated group H such that $G_\pi \cong H_\pi$ for every finite set of primes π is called the **restricted genus** of G.

Assigning a natural number $n(G)$ to a $\mathcal{X}_0$-group G

Let n_1 be the exponent of T_G,
let n_2 be the exponent of the group $\mathrm{Aut}(T_G)$, and
let n_3 be the exponent of the torsion subgroup of the centre of G.
Consider $n(G) = n_1 n_2 n_3$.
$n = n(G)$ has the property that the subgroup $G^{(n)} = \langle g^n \, : \, g \in G \rangle$ of G belongs to the centre of G and $G/G^{(n)}$ is a finite group.

Assigning a natural number $n(G)$ to a $\mathcal{X}_0$-group G

Let $\pi = \{p : \ p \text{ is a prime and } p | n(G)\}$. Then the short exact sequence

$$1 \to G^{(n)} \to G \to G/G^{(n)} \to 1$$

determines G as an extension of a π'-torsion-free finitely generated abelian group $G^{(n)}$ by a π-torsion group $G/G^{(n)}$. From [**?**, Proposition 3.1], it follows that the π-localization homomorphism

$$G \to G_\pi$$

is injective.

Group structure on the noncancellation set

Witbooi in [16] shows that the non-cancellation set of a $\mathcal{X}_0$-group G has a group structure and there is an epimorphism

$$\zeta : \mathbb{Z}_n^*/\pm 1 \to \chi(G)$$

, where $n = n(G)$. [16] P.J. Witbooi, Generalizing the Hilton-Mislin genus group, *J. Algebra* **239** (2001), no. 1, 327-339.

Group structure on the noncancellation set

1. For a nilpotent $\mathcal{X}_0$-group G, Warfield in [15] shows that

$$\chi(G) \cong \mathcal{G}(G)$$

.

2. O'Sullivan in [13] shows that for a $\mathcal{X}_0$-group G,

$$\chi(G) \cong \tau_f(G)$$

.

[13] N. O'Sullivan, Genus and cancellation, *Comm. Algebra* **28** (2000), no. 7, 3387-3400.
[15] R. Warfield, Genus and cancellation for groups with finite commutator subgroup, *J. Pure Appl. Algebra* **6** (1975) 125-132.

Homomorphisms between non-cancellation groups

Existence of homomorphisms

For a semidirect product $H = \mathbb{Z}_m \rtimes_\omega \mathbb{Z}$, the authors in [5] showed that there is a well-defined surjective homomorphism

$$\Gamma : \chi(H) \to \chi(H^r)$$

given by $[K] \to [K \times H^{r-1}]$ where K is a group such that $K \times \mathbb{Z} \cong H \times \mathbb{Z}$ and r is a natural number. Thus, in order to compute the group $\chi(H^r)$ one needs only to compute the kernel of the homomorphism Γ.

[5]A. Fransman and P. Witbooi, Non-cancellation sets of direct powers of certain metacyclic groups, *Kyungpook Math. J.* **41** (2001), no. 2, 191-197.

Computation of $\chi(G_1 \times G_2)$

[16] P.J. Witbooi, Generalizing the Hilton-Mislin genus group, *J. Algebra* **239** (2001), no. 1, 327-339.

Description

Witbooi in [16] notice that for any $\mathcal{X}_0$-groups G_1 and G_2 and for groups K belonging to $\chi(G_1)$, the rule $K \mapsto K \times G_2$ induces a well-defined function $\theta : \chi(G_1) \to \chi(G_1 \times G_2)$ which is an epimorphism.

Category of $\mathcal{X}_0$-groups under a finite group F

Let us fix a finite group F. Let Grp_F be the category of groups under F. Here we mean that the objects of Grp_F are group homomorphisms $\varphi : F \to G$.
Given another object $\varphi_1 : F \to G_1$, a morphism in Grp_F corresponds to a group homomorphism $\alpha : G \to G_1$ such that $\alpha \circ \varphi = \varphi_1$.
For a set of primes π, the π-localization of an object $\varphi : F \to G$ will be the object $\varphi_\pi : F \to G_\pi$. Then localization is an endofunctor of Grp_F.

Category of $\mathcal{X}_0$-groups under a finite group F

Let $\mathcal{X}_F$ be the full subcategory of $\mathcal{X}_0$-groups under F. We can define the restricted genus
$\Gamma_f(\varphi) = \{[\psi] \mid \psi_\pi \text{ is isomophic to } \varphi_\pi\}$
If F is the trivial group, then $\mathcal{X}_F$ can be identified with the class $\mathcal{X}_0$ of groups.
In line with [16] and in analogy with $\mathcal{X}_0$-groups we shall write
$\Gamma_f(\phi) = \chi(G, \phi)$.

[16] P.J. Witbooi, Generalizing the Hilton-Mislin genus group, *J. Algebra* **239** (2001), no. 1, 327-339.

Category of $\mathcal{X}_0$-groups under a finite group F

Theorem [11, Theorem 2.3]

Let (L, l) be an object representing a member of $\chi(G, h)$. Then there exist a subgroup J of G with $[G : J]$ finite and $[G : J]$ relatively prime to n, such that in Grp_F the object $F \to J$ is isomorphic to (L, l).

[11]J.C. Mba and P.J. Witbooi, Induced morphisms between localization genera of groups, *Algebra Colloquium*, 21:2 (2014) 285-294.

Homomorphisms between non-cancellation groups

Existence of homomorphisms

Let F be a finite group and consider the homomorphism $h : F \to G$.
In [11], a group structure is defined on $\chi(G, h)$ and an epimorphism

$$\zeta : (\mathbb{Z}/n)^*/\pm 1 \to \chi(G, h)$$

is established.
It is also shown that there exist natural epimorphisms

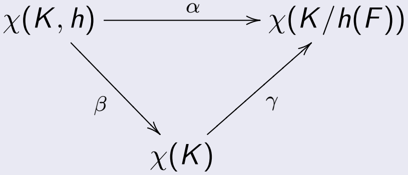
$$\chi(G, h) \to \chi(G/h(F)) \text{ and } \chi(G, h) \to \chi(G, h \circ i))$$

.

[11]J.C. Mba and P.J. Witbooi, Induced morphisms between localization genera of groups, *Algebra Colloquium*, 21:2 (2014) 285-294.

Non-existence of homomorphisms

In [11], computation methods of $\chi(G, h)$ in the special case G is a semidirect product $T \rtimes_\omega \mathbb{Z}^k$ are used in a very particular example to provide a concrete computation of $\chi(G, h)$. It is used to show that there doesn't exist any homomorphism γ to make the following diagram commutative



[11]J.C. Mba and P.J. Witbooi, Induced morphisms between localization genera of groups, *Algebra Colloquium*, 21:2 (2014) 285-294.

Homomorphisms between localization genera

Notation

Fix any $m \in \mathbb{N}$. Let $X(m) = \{u \in \mathbb{N} | \ (u, m) = 1\}$. Now consider any $G \in \mathcal{X}_0$ and let $n = n(G)$. Let $Y(G, h)$ be the set of all $u \in X(n)$ for which there exists a subgroup J of G with $[G : J] = u$ and such that the object (J, h_J) represents a member of $\chi(G, h)$. Here h_J is the induced homomorphism obtained from h by restriction of the codomain. For each $u \in Y(G, h)$, let us choose a subgroup G_u of G such that $T_G \subseteq G_u$ and $[G : G_u] = u$. Let $h_u : F \to G_u$ be the induced homomorphism defined by $h_u : x \mapsto h(x)$. Now let us denote the isomorphism class of the object h_u of $\mathcal{X}_F$ by $[G_u, h_u]$. Then we obtain a function $\xi : Y(G, h) \to \chi(G, h)$. Let $Y^*(G, h)$ denote the image of $Y(G, h)$ in $\mathbb{Z}_n^*$.

Homomorphisms between localization genera

Theorem [11, Theorem 2.5]

a) $Y^*(G,h)$ is a subgroup of $\mathbb{Z}_n^*$.

b) The function ξ induces a (well-defined) function

$$\zeta : Y^*(G,h)/\pm 1 \to \chi(G,h).$$

c) The fibre $\zeta^{-1}[G,h]$ of ζ over $[G,h]$ is a subgroup of $Y^*(G,h)/\pm 1$.

d) For any $[K,k] \in \chi(G,h)$, $\zeta^{-1}[K,k]$ is a coset of $\zeta^{-1}[G,h]$.

[11]J.C. Mba and P.J. Witbooi, Induced morphisms between localization genera of groups, *Algebra Colloquium*, 21:2 (2014) 285-294.

Homomorphisms between localization genera

Theorem

Let (G_1, h_1) and (G_2, h_2) be such that $n_1 = n(G_1)$ and $n_2 = n(G_2)$ are relatively prime. Then,

1. There is a homomorphism

$$\alpha : Y^*(G_1, h_1)/\pm 1 \to Y^*(G_2, h_2)/\pm 1$$

   defined by $u \mapsto n_1^{\lfloor ln(u) \rfloor}$.

2. There are homomorphisms φ and β such that the following diagram is commutative.

$$\begin{array}{ccc} Y^*(G_1, h_1)/\pm 1 & \xrightarrow{\xi_1} & \chi(G_1, h_1) \\ & \searrow^{\beta} & \downarrow \varphi \\ & & \chi(G_2, h_2) \end{array}$$

Homomorphisms between localization genera

Corollary

Let (G_1, h_1) and (G_2, h_2) be such that $n_1 = n(G_1)$ and $n_2 = n(G_2)$ are relatively prime. Let φ and β the following homomorphisms

$$\begin{array}{ccc} Y^*(G_1, h_1)/\pm 1 & \xrightarrow{\xi_1} & \chi(G_1, h_1) \\ & \searrow^{\beta} & \downarrow \varphi \\ & & \chi(G_2, h_2) \end{array}$$

If β is surjective, then φ is an epimorphism.

Homomorphisms between localization genera

Proposition [16, Proposition 6.1]

. Suppose that we have groups A, B and C together with a homomorphism $\beta : A \to C$ and a surjective group homomorphism $\gamma : A \to B$. If $\alpha : B \to C$ is a function (between sets) such that $\alpha \circ \gamma = \beta$, then α is a homomorphism. Moreover, if β is surjective, then α is also surjective.

[16] P.J. Witbooi, Generalizing the Hilton-Mislin genus group, *J. Algebra* **239** (2001), no. 1, 327-339.

THANK YOU

- G. Baumslag, Lecture notes on nilpotent groups, *Regional Conference Series in Mathematics*, No. 2, *American Mathematical Society, Providence, R.I.* 1971.
- G. Baumslag, Some remarks on nilpotent groups with roots, *Proc. Amer. Math. Soc.* **12** (1961) 262-267.
- A.K. Bousfield and D. M. Kan, Homotopy limits, completions and localizations, *Lecture Notes in Mathematics*, Vol. 304. Springer-Verlag, Berlin-New York, 1972.
- [1] C. Casacuberta and P. Hilton, Calculating the Mislin genus for a certain family of nilpotent groups, *Comm. Algebra* **19** (1991), no. 7, 2051-2069.
- [2]A. Fransman and P. Witbooi, Non-cancellation sets of direct powers of certain metacyclic groups, *Kyungpook Math. J.* **41** (2001), no. 2, 191-197.
- [3]P. Hilton and C. Schuck, On the structure of nilpotent groups of a certain type, *Topol. Methods Nonlinear Anal.* **1** (1993), no. 2, 323-327.