On the pronormality of subgroups of odd indices in finite simple groups

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This talk based on joint works with
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DEFINITION (Ph. Hall). A subgroup H of a group G is pronormal in G if H and H^g are conjugate in $\langle H, H^g \rangle$ for every $g \in G$.

AGREEMENT. Further we consider finite groups only.

EXAMPLES. The following subgroups are pronormal in finite groups:

- Normal subgroups;
- Maximal subgroups;
- Sylow subgroups;
- Sylow subgroups of normal subgroups.

PROPOSITION. Let $A \trianglelefteq G$ and $H \leq A$.
The following statements are equivalent:

(1) H is pronornal in G;
(2) H is pronormal in A and $G = AN_G(H)$.

WHY IS IT INTERESTING?

DEFINITION (L. Babai). A group G is called a CI-group if between every two isomorphic relational structures on G (as underlying set) which are invariant under the group $G_R = \{g_R \mid g \in G\}$ of right multiplications

$$g_R : x \mapsto xg,$$

there exists an isomorphism which is at the same time an automorphism of G.

THEOREM (L. Babai, 1977). G is a CI-group if and only if G_R is pronormal in $Sym(G)$.

COROLLARY. If G is a CI-group then G is abelian.

THEOREM (P. Pálfy, 1987). G is a CI-group if and only if $|G| = 4$ or G is cyclic of order n such that $(n, \varphi(n)) = 1$.

WHY IS IT INTERESTING?

A conjugacy class of a pronormal subgroup is an example of a locally conjugate collection of subgroups.

DEFINITION (M. Aschbacher and M. Hall, Jr.) A collection Δ of subgroups of a group G is said to be locally conjugate if

(1) $\Delta = \Delta^G$ (i. e. $A \in \Delta \Rightarrow A^g \in \Delta$ for all $g \in G$);

(2) $G = \langle \Delta \rangle$;

(3) if $A, B \in \Delta$ then either $[A, B] = 1$ or A and B are conjugate in $\langle A, B \rangle$.

If D is a class of odd transpositions of a group (for instance, 3-transpositions) then $\Delta = \{\langle d \rangle \mid d \in D\}$ is locally conjugate.

If H is pronormal in G then H^G is locally conjugate in $\langle H^G \rangle$.

WHY IS IT INTERESTING?

H is pronormal in G if H and H^g are conjugate in $\langle H, H^g \rangle$ for every $g \in G$.

THEOREM (Ch. Praeger, 1984). Let G be a transitive permutation group on a set Ω of n points, and let K be a nontrivial pronormal subgroup of G. Suppose that K fixes exactly f points of Ω. Then

(a) $f \leq \frac{1}{2}(n-1)$, and

(b) if $f = \frac{1}{2}(n-1)$ then K is transitive on its support in Ω, and either $G \geq A_n$, or $G = GL(d, 2)$ acting on the $n = 2^d - 1$ nonzero vectors, and K is the pointwise stabilizer of a hyperplane.

CONJECTURE

H is pronormal in G if H and H^g are conjugate in $\langle H, H^g \rangle$ for every $g \in G$.

A group G is simple if G does not contain proper normal subgroups.

QUESTION. What are pronormal subgroups of finite simple groups?

DEFINITION. H is a Hall subgroup of G if $(|H|, |G : H|) = 1$.

THEOREM (E. Vdovin and D. Revin, 2012). Every Hall subgroup is pronormal in every finite simple group.

CONJECTURE (E. Vdovin and D. Revin, 2012). The subgroups of odd index (= the overgroups of Sylow 2-subgroups) are pronormal in finite simple groups.

ON THE FINITE SIMPLE GROUPS

A group G is simple if G does not contain proper normal subgroups.

With respect to the Classification of Finite Simple Groups, finite simple groups are:

- Cyclic groups C_p, where p is a prime;
- Alternating groups $Alt(n)$ for $n \geq 5$;
- Classical groups: $PSL_n(q) = L_n(q)$,
  $PSU_n(q) = U_n(q) = PSL_n^-(q) = L_n^-(q)$,
  $PSp_{2n}(q) = S_{2n}(q)$, $P\Omega_n(q) = O_n(q)$ (n is odd),
  $P\Omega_n^+(q) = O_n^+(q)$ (n is even),
  $P\Omega_n^-(q) = O_n^-(q)$ (n is even);
- Exceptional groups of Lie type:
  $E_8(q)$, $E_7(q)$,
  $E_6(q)$, ${}^2E_6(q) = E_6^-(q)$,
  ${}^3D_4(q)$, $F_4(q)$, ${}^2F_4(q)$,
  $G_2(q)$, ${}^2G_2(q) = Re(q)$ (q is a power of 3),
  ${}^2B_2(q) = Sz(q)$ (q is a power of 2);
- 26 sporadic groups.

H is pronormal in G if H and H^g are conjugate in $\langle H, H^g \rangle$ for every $g \in G$.

PROPOSITION 1. Let G be a group, $S \leq H \leq G$ and S be a pronormal (for example, Sylow) subgroup of G. Then the following conditions are equivalent:

(1) H is pronormal in G;
(2) H and H^g are conjugate in $\langle H, H^g \rangle$ for every $g \in N_G(S)$.

REMARK. Let G be a group, $H \leq G$ and S be a pronormal subgroup of G. If $N_G(S) \leq H$ then H is pronormal in G.

LEMMA 1 (A. Kondrat'ev, 2005). Let G be a finite nonabelian simple group and $S \in Syl_2(G)$. Then $N_G(S) = S$ excluding the following cases:

(1) $G \cong J_2,\ J_3,\ Suz$ or HN and $|N_G(S) : S| = 3$;

(2) $G \cong {}^2G_2(3^{2n+1})$ or J_1 and $N_G(S)/S \cong 7 \rtimes 3$;

(3) G is a group of Lie type over field of characteristic 2 and $N_G(S)$ is a Borel subgroup of G;

(4) $G \cong PSL_2(q)$ where $3 < q \equiv \pm 3 \pmod 8$ and $N_G(S) \cong A_4$;

(5) $G \cong PSp_{2n}(q)$, where $n \geq 2$, $q \equiv \pm 3 \pmod 8$, $n = 2^{s_1} + \cdots + 2^{s_t}$ for $s_1 > \cdots > s_t \geq 0$ and $N_G(S)/S$ is the elementary abelian group of order 3^t;

(6) $G \cong PSL_n^\eta(q)$, where $n \geq 3$, $\eta = \pm$, q is odd, $n = 2^{s_1} + \cdots + 2^{s_t}$ for $s_1 > \cdots > s_t > 0$ and $N_G(S) \cong S \times C_1 \times \cdots \times C_{t-1}$, where $C_1, \ldots C_{t-2}, C_{t-1}$ are cyclic subgroup of orders $(q-\eta 1)_{2'}, \ldots, (q-\eta 1)_{2'}, (q-\eta 1)_{2'}/(q-\eta 1, n)_{2'}$ respectively;

(7) $G \cong E_6^\eta(q)$ where $\eta = \pm$ and q is odd and $|N_G(S) : S| = (q-\eta 1)_{2'}/(q-\eta 1, 3)_{2'} \neq 1$.

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PRONORMAL SUBGROUPS

H is pronormal in G if H and H^g are conjugate in $\langle H, H^g \rangle$ for every $g \in G$.

THEOREM 1 (A. Kondrat'ev, N.M., D. Revin, 2015). All subgroups of odd index are pronormal in the following finite simple groups:

(1) $Alt(n)$, where $n \geq 5$;
(2) sporadic groups;
(3) groups of Lie type over fields of characteristic 2;
(4) $PSL_{2^n}(q)$;
(5) $PSU_{2^n}(q)$;
(6) $PSp_{2n}(q)$, where $q \not\equiv \pm 3 \pmod 8$;
(7) $P\Omega_n^{\varepsilon}(q)$, where $\varepsilon \in \{+, -, \text{epmty symbol}\}$;
(8) exceptional groups of Lie type not isomorphic to $E_6(q)$ or ${}^2E_6(q)$.

H is pronormal in G if H and H^g are conjugate in $\langle H, H^g \rangle$ for every $g \in G$.

PROBLEM. Are the subgroups of odd index pronormal in the following finite simple groups:

(1) $PSL_n(q)$, where $n \neq 2^w$ and q is odd;
(2) $PSU_n(q)$, where $n \neq 2^w$ and q is odd;
(3) $PSp_{2n}(q)$, where $q \equiv \pm 3 \pmod 8$;
(4) exceptional groups of Lie type $E_6(q)$ and ${}^2E_6(q)$, where q is odd?

COUNTEREXAMPLE TO THE CONJECTURE

H is pronormal in G if H and H^g are conjugate in $\langle H, H^g \rangle$ for every $g \in G$.

Let $q \equiv \pm 3 \pmod 8$ be a prime power and n be a positive integer. It's well known, Sylow 2-subgroup S of a group $T = Sp_2(q) = SL_2(q)$ is isomorphic to Q_8 and $N_T(S) \cong SL_2(3) = Q_8 : 3$. We have

$H = Q_8 \wr Sym(3n) \leq X = SL_2(3) \wr Sym(3n) \leq Y = Sp_2(q) \wr Sym(3n) \leq G = Sp_{6n}(q)$.

The index $|G : H|$ is odd and $H/Z(G)$ is a nonpronormal subgroup of odd index in $G/Z(G) = PSp_{6n}(q)$.

LEMMA 1 (A. Kondrat'ev, 2005). Let G be a finite nonabelian simple group and $S \in Syl_2(G)$. Then $N_G(S) = S$ excluding the following cases:

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CLASSIFICATION PROBLEM

H is pronormal in G if H and H^g are conjugate in $\langle H, H^g \rangle$ for every $g \in G$.

PROBLEM. Classify finite simple groups in which all subgroups of odd index are pronormal.

THEOREM 2 (A. Kondrat'ev, N.M., D. Revin, 2016). Let $G = PSp_n(q)$, where $q \equiv \pm 3 \pmod 8$ and $n \notin \{2^m, 2^m(2^{2k}+1) \mid m, k \in \mathbb{N}\}$. Then G contains a nonpronormal subgroup of odd index.

THEOREM 3 (A. Kondrat'ev, N.M., D. Revin, 2016). Let $G = PSp_n(q)$. Then every subgroup of odd index is pronormal in G if and only if one of the following conditions holds:

(1) $q \not\equiv \pm 3 \pmod 8$;

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SKETCH OF PROOF

$G = PSp_n(q)$, where $q \equiv \pm 3 \pmod 8$ and $n \in \{2^m, 2^m(2^{2k}+1) \mid m, k \in \mathbb{N}\}$;

$H \leq G$ and $|G : H|$ is odd;

$S \in Syl_2(G)$ such that $S \leq H$;

$g \in N_G(S)$ and $K = \langle H, H^g \rangle$;

$K = G \Rightarrow H$ and H^g are conjugate in $\langle H, H^g \rangle$;

$K \neq G \Rightarrow \exists M$: $K \leq M$ and M is maximal in G;

Do we know M?

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SOME TOOLS

Let $m = \sum_{i=0}^{\infty} a_i \cdot 2^i$ and $n = \sum_{i=0}^{\infty} b_i \cdot 2^i$, where $a_i, b_i \in \{0, 1\}$.
We write $m \preceq n$ if $a_i \leq b_i$ for every i and $m \prec n$ if, in addition, $m \neq n$.

THEOREM (N.M., 2008). Maximal subgroups of odd index in $Sp_{2n}(q) = Sp(V)$, where $n > 1$ and q is odd are the following:

(1) $Sp_{2n}(q_0)$, where $q = q_0^r$ and r is an odd prime;
(2) $Sp_{2m}(q) \times Sp_{2(n-m)}(q)$, where $m \prec n$;
(3) $Sp_{2m}(q) \wr Sym(t)$, where $n = mt$ and $m = 2^k$;
(4) $2_+^{1+4}.Alt(5)$, where $n = 2$ and $q \equiv \pm 3 \pmod 8$ is a prime.

DIFFICULTIES

H is pronormal in G if H and H^g are conjugate in $\langle H, H^g \rangle$ for every $g \in G$.

Let $\mathbb{X}_2$ be the class of all finite simple groups with self-normalized Sylow 2-subgroups,
$\mathbb{Y}_2$ be the class of all finite groups in which the subgroups of odd index are pronormal.

Let G and K be finite groups, $H \leq G$ and $A \trianglelefteq G$. Then

(1) $G \in \mathbb{Y}_2 \Rightarrow G/A \in \mathbb{Y}_2$

(2) $G \in \mathbb{Y}_2 \not\Rightarrow H \in \mathbb{Y}_2$

(3) $G \in \mathbb{Y}_2 \not\Rightarrow A \in \mathbb{Y}_2$

(4) $G, K \in \mathbb{Y}_2 \not\Rightarrow G \times K \in \mathbb{Y}_2$

even for finite simple groups!

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SOME TOOLS TO WIN DIFFICULTIES

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THEOREM 4 (W. Guo, N.M., D. Revin, 2016-2017). Let G be a finite group, $A \trianglelefteq G$, $A \in \mathbb{Y}_2$, and $G/A \in \mathbb{X}_2$. Let T be a Sylow 2-subgroup of A. Then the following conditions are equivalent:

(1) $G \in \mathbb{Y}_2$;
(2) $N_G(T)/T \in \mathbb{Y}_2$.

SOME TOOLS TO WIN DIFFICULTIES

H is pronormal in G if H and H^g are conjugate in $\langle H, H^g \rangle$ for every $g \in G$.

If $m = \sum_{i=0}^{\infty} a_i \cdot 2^i$ and $n = \sum_{i=0}^{\infty} b_i \cdot 2^i$, where $a_i, b_i \in \{0, 1\}$.

We write $m \preceq n$ if $a_i \leq b_i$ for every i and $m \prec n$ if, in addition, $m \neq n$.

THEOREM 5 (W. Guo, N.M., D. Revin, 2016-2017). Let A be a finite abelian group and $G = \prod_{i=1}^{t}(A \wr Sym(n_i))$, where all the wreath products are natural permutation. Then all subgroups of odd index are pronormal in G if and only if for any positive integer m, if $m \prec n_i$ for some i then $h.c.f.(|A|, m)$ is a power of 2.

PRONORMAL SUBGROUPS

H is pronormal in G if H and H^g are conjugate in $\langle H, H^g \rangle$ for every $g \in G$.

THEOREM 3 (A. Kondrat'ev, N.M., D. Revin, 2016).
Let $G = PSp_n(q)$. Then every subgroup of odd index is pronormal in G if and only if one of the following conditions holds:

(1) $q \not\equiv \pm 3 \pmod 8$;
(2) $n \in \{2^m, 2^m(2^{2^k}+1) \mid m, k \in \mathbb{N}\}$.

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Let G be an exceptional group of Lie type $E_6^{\varepsilon}(q)$, where q is odd and $\varepsilon \in \{+, -\}$. Then every subgroup of odd index is pronormal in G if and only if 9 does not divide $q - \varepsilon 1$.

PROBLEM. Are all subgroups of odd index pronormal in $PSL_n(q) = L_n^+(q)$ and $PSU_n(q) = L_n^-(q)$, where $n \neq 2^w$ and q is odd?

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If $m = \sum_{i=0}^{\infty} a_i \cdot 2^i$ and $n = \sum_{i=0}^{\infty} b_i \cdot 2^i$, where $a_i, b_i \in \{0, 1\}$.

We write $m \preceq n$ if $a_i \leq b_i$ for every i and $m \prec n$ if, in addition, $m \neq n$.

CONJECTURE. Let $G = L_n^{\varepsilon}(q)$, where q is odd and $\varepsilon \in \{+, -\}$. All subgroups of odd index are pronormal in G if and only if for any positive integer m, if $m \prec n$ then $h.c.f.(m, q^{1+\varepsilon 1}(q - \varepsilon 1))$ is a power of 2.

WHERE CAN WE APPLY THE RESULTS?

H is pronormal in G if H and H^g are conjugate in $\langle H, H^g \rangle$ for every $g \in G$.

THEOREM (Ch. Praeger, 1984). Let G be a transitive permutation group on a set Ω of n points, and let K be a nontrivial pronormal subgroup of G. Suppose that K fixes exactly f points of Ω. Then

(a) $f \leq \frac{1}{2}(n-1)$, and

(b) if $f = \frac{1}{2}(n-1)$ then K is transitive on its support in Ω, and either $G \geq A_n$, or $G = GL(d,2)$ acting on the $n = 2^d - 1$ nonzero vectors, and K is the pointwise stabilizer of a hyperplane.

If G is simple, $|\Omega|$ is odd, and $x \in \Omega$ then G_x is usually pronormal in G, and we wish to know all the exceptions.

Thank you for your attention!