On recognizing finite simple groups by element orders in the class of all groups

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We are generally interested in the following question:

$\omega(G)$ given $\Rightarrow G$?

if $\omega(G)$ is given, what can we say about G?

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In other words: does every finite set of elements from G generate a finite subgroup?
And general answer is
no:
there exists finitely generated infinite group G of period n (so take $\omega(G) = \{m|n\}$) for
$n > 665$ odd (Adyan 1975)
$n > 8000$ even (Lysenok 1996)

Known results

$\{1,5\} \neq \omega(G) \subseteq \{1,2,3,4,5,6\}$

And the answer is yes, a corresponding group G is locally finite, provided that element orders of G are not greater than 6, and G is not a group of period 5.

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The list of contributors is quite long:
W. Burnside 1902 (period 3);
B. Neumann 1937 ($\omega = \{1,2,3\}$);
I. Sanov 1940 (period 4);
M. Hall 1958 (period 6);
M. Newman 1979 ($\omega = \{1,2,5\}$);
N. Gupta and V. Mazurov 1999 + E. Jabara 2004 ($\omega(G) \subset \{1,2,3,4,5\}$);
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So basically each ω requires individual attention.

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Another remark:

2. If we additionally assume that G is finite then the question is specifically interesting for $\omega(G) = \omega(H)$, where H is some non-abelian finite simple group, when very often we may conclude that $H \simeq G$.

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$\Rightarrow$ naturally we face the corresponding Burnside problem
But modulo what is already known for finite groups usually this is the major problem that should be solved.

Recognizable

The following groups are known to be recognizable by their sets of element orders (spectra) in the class of all groups:

$L_2(2^m)$ A. Zhurtov, V. Mazurov 1999
$L_2(7) \simeq L_3(2)$ D. Lytkina, A. Kuznetsov 2007
M_{10} (not simple) E. Jabara, D. Lytkina, A. M. 2014
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In some sense it is more pleasant to have a business here with simple groups. And I want to try to explain why by demonstrating the role of normal subgroups, centralizers of involution, etc. in the proof.

Warning

Note that there are finite simple groups, which are recognizable by spectrum in the class of finite groups, but are not recognizable in the class of all groups.

V. Mazurov, A. Olshanskiy, A. Sozutov 2016

Let $m = 2^{10}k \geq 2^{49}$ be an integer such that $q = m + \epsilon$ is a power of prime for $\epsilon \in \{1, -1\}$. Then $L_2(q)$ is not recognizable by spectrum in the class of all groups.

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So it would be nice to be able to construct locally finite normal subgroups and do the reduction. There is a good candidate for that — $O_2(G)$, provided it is nontrivial: because for groups G in the list $O_2(G)$ is a group of period 4, and hence locally finite by Sanov's theorem.

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But what can we "reduce" (factor out) this way? Our group has an involution, and two involutions in a periodic group always generate a finite (dihedral) group. And there is a well-known criterion for finite G ensuring that involution i is in $O_2(G)$ in terms of subgroups generated by two involutions:

Baer-Suzuki theorem

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And in general the answer is <u>no</u>.
Corresponding example for large periods $n = 2^m \geq 2^{48}$ was constructured by V. Mazurov, A. Olshanskiy, A. Sozutov in the same work.

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However for groups in the list — yes — the theorem holds.

A. M. 2016

If G is a group of period $n = 4k$, where k is odd, i is an involution, and any two elements from i^G generate a 2-subgroup, then $\langle i^G \rangle$ is also 2-subgroup.

(2,3)-generated subgroups

Further we take an involution and an element of order 3 from G and list all possibilities for a subgroup that they generate:
S_3
A_4
A_5
$L_2(7)$
homomorphic images of Frobenius groups $(C_k \times C_6) \rtimes C_6$
$\dots$

It is possible to construct this list using coset enumeration in GAP: groups are small.

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Strategy

Further strategy is to reduce the list of possibilities mod statement that G contains a non-abelian finite simple subgroup H_0.

Explaining strategy

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And first thing we want to remove from the list of possibilities is A_4. Here we use Baer-Suzuki theorem to deduce that either $O_2(A_4)$ is in $O_2(G)$ and so can be factored-out, or an involution from A_4 must invert some nontrivial element of odd order, and using local analysis and coset enumeration we obtain f.s.subgroup H_0.

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After that the situation simplifies.