On a finiteness condition on non-abelian subgroups

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Basic Problem

Let G be a **group** and let $\mathfrak{M}$ be a **family of subgroups** of G.

Main Problem

Obtain information about the structure of G

by looking at properties concerning $\mathfrak{M}$.

Basic Problem

Let G be a (possibly infinite) **group** and let $\mathfrak{M}$ be a **family of subgroups** of G.

Main Problem

Find information about the structure of G assuming that $\mathfrak{M}$ satisfies a finiteness condition.

Background - $\mathfrak{L}(G)$

Let G be a (possibly infinite) group.

Example

Let $\mathfrak{M} = \mathfrak{L}(G)$ be the family of all subgroups of G.
Then

$$\mathfrak{L}(G) \text{ is finite} \Leftrightarrow G \text{ is finite}.$$

There are many well-known classical results

about **classes of groups** G with

$\mathfrak{L}(G) \in \mathcal{M}ax$ or $\mathfrak{L}(G) \in \mathcal{M}in$.

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Background - $\mathfrak{L}(G)$

Theorem

Let G be a soluble group.

$\mathfrak{L}(G)$ has $\mathcal{M}ax$

if and only if G is polycyclic.

Definition

A group G is said to be **polycyclic** if it has a finite series whose factors are cyclic

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Theorem, (S.N. Černikov)

Let G be a soluble group.

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if and only if

G has an abelian subgroup A of finite index such that

A is direct product of finitely many quasi-cyclic groups.

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Definitions - $\mathfrak{L}_{\mathcal{P}}(G)$, $\mathfrak{L}_{non-\mathcal{P}}(G)$

Let $\mathcal{P}$ be a group theoretical property.

Definitions

Denote by $\mathfrak{L}_{\mathcal{P}}(G)$

the family of all subgroups H of G such that H has $\mathcal{P}$

and by

$\mathfrak{L}_{non-\mathcal{P}}(G)$

the family of subgroups H of G such that H does not have $\mathcal{P}$.

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Background - $\mathfrak{L}_{\mathcal{P}}(G)$, $\mathfrak{L}_{non-\mathcal{P}}(G)$

Example

If $\mathfrak{L}_{non-\mathcal{P}}(G) = \{G\}$,
then **every** proper subgroup of G has $\mathcal{P}$.

Groups G with **finiteness conditions**
on $\mathfrak{L}_{\mathcal{P}}(G)$ or on $\mathfrak{L}_{non-\mathcal{P}}(G)$
for various properties $\mathcal{P}$ have been studied by many authors.

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Background - $\mathfrak{L}_{ab}(G)$, $\mathfrak{L}_{non-ab}(G)$

Example

If $\mathcal{P} = ab$ is the property to be **abelian**, then

$\mathfrak{L}_{ab}(G)$ is finite $\Leftrightarrow$ G is finite.

Remark

Abelian and Minimal non-abelian groups are groups G with $\mathfrak{L}_{non-ab}(G)$ **finite**.

Background - $\mathfrak{L}_{ab}(G)$

Let $\mathcal{P} = ab$ be the property to be **abelian**.

Groups G in which $\mathfrak{L}_{ab}(G)$, ordered by inclusion,

has $\mathcal{M}ax$ or $\mathcal{M}in$

have been firstly studied respectively by

A.I. Mal'cev in 1956 and **O.J. Schmidt** in 1945.

A.I. Mal'cev, On certain classes of infinite soluble groups, *Mat. Sb.* **28** (1951), 567-588 (Russian), *Amer. Math. Soc. Transl.* (2) **2** (1956), 1-21.

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L.A. Kurdachenko and **D.I. Zaicev** in 1991.

L.A. Kurdachenko, D.I. Zaicev, Groups with the maximum condition for non-abelian subgroups, *Ukrain. Mat. Zh.* **43** (1991), 925-930 (Russian), English transl., *Ukrainian Math. J.* **43** (1991), 863-868.

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Background - $\mathfrak{L}_{non-ab}(G)$

Groups in which $\mathfrak{L}_{non-ab}(G)$

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S.N. Černikov in 1964 and 1967.

S.N. Černikov, Infinite groups with prescribed properties of their systems of infinite subgroups, *Dokl. Akad. Nauk SSSR*, **159** (1964) 759-760 (Russian), *Soviet Math. Dokl.*, **5** (1964) 1610-1611.

S.N. Černikov, Groups with given properties of systems of infinite subgroups, *Ukrain. Mat. Ž*, **19** (1967) 111-131 (Russian), *Ukrainian Math. J.*, **19** (1967) 715-731.

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Tarski monsters

Remark

Some rather exotic examples of groups
can be found in studying this type of problems.

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There exist infinite simple groups in which
every proper non-trivial subgroup has order a fixed prime p,
the so-called **Tarski monsters**.

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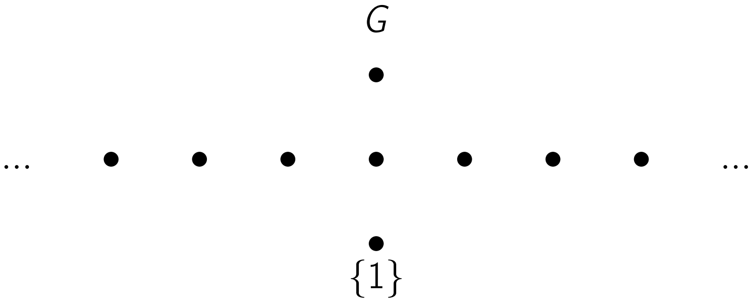

A. Yu Ol'shanskii, *Geometry of defining relations in groups*, Mathematics and its Applications, vol.**70** Kluwer Academic Publishers, Dordrecht, 1989.

Remark

Tarski monsters are groups

in which

$\mathfrak{L}(G)$ has $\mathcal{M}ax$, $\mathcal{M}in$,

$\mathfrak{L}_{ab}(G)$ has $\mathcal{M}ax$, $\mathcal{M}in$,

$\mathfrak{L}_{non-ab}(G)$ **is finite.**

Some sample results

Theorem, (B.I. Plotkin, 1956)

Let G be a radical group.

$\mathfrak{L}_{ab}(G)$ has $\mathcal{M}in$,

if and only if

G is soluble and $\mathfrak{L}(G)$ has $\mathcal{M}in$.

$\mathfrak{L}_{non-ab}(G)$ has $\mathcal{M}in$, if and only if either G is abelian or G is soluble and $\mathfrak{L}(G)$ has $\mathcal{M}in$.

Definition

A group G is called **radical** if there exists an ascending series of G with locally nilpotent factors.

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A group G is called **radical** if there exists
an ascending series of G with locally nilpotent factors.

Paper

L.A. Kurdachenko, P. Longobardi, M. M., I.Ya Subbotin

Groups with finitely many

isomorphic classes of non-abelian subgroups

submitted.

A new problem

We study

another finiteness condition on

$\mathfrak{L}_{\mathcal{P}}(G)$ and $\mathfrak{L}_{non-\mathcal{P}}(G)$.

Definition

Let G be a **group** and let $\mathfrak{M}$ be a **family of subgroups** of G.

Definition

Consider the equivalence relation in $\mathfrak{M}$ given by

$H \simeq K$, with $H, K \in \mathfrak{M}$.

Call isomorphic type **Itype**$\mathfrak{M}$ of $\mathfrak{M}$

any set of representatives of all equivalence classes in $\mathfrak{M}$.

Problem

Let G be a **group** and let $\mathfrak{M}$ be a **family of subgroups** of G.

Definitions

Consider the equivalence relation in $\mathfrak{M}$ given by
$H \simeq K$, with $H, K \in \mathfrak{M}$.

Call the isomorphic type **Itype**$\mathfrak{M}$ of $\mathfrak{M}$
any set of representatives of all equivalence classes in $\mathfrak{M}$.

We study groups G in which
Itype$\mathfrak{M}$ is **finite**.

First remarks - $|\textbf{Itype}\mathfrak{L}(G)|$

Let G be a group.

Remark

If G is non-trivial, then $G, \{1\} \in \textbf{Itype}\mathfrak{L}(G)$.
Thus

$$|\textbf{Itype}\mathfrak{L}(G)| \geqslant 2.$$

Proposition

$|\textbf{Itype}\mathfrak{L}(G)| = 2 \Leftrightarrow$ either $|G|$ a prime or G infinite cyclic.

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Problem

What about $|\mathbf{Itype}\mathfrak{L}_{ab}(G)|$?

Remark

Obviously, if $G \neq \{1\}$, then $\{1\}, < x > \in \mathbf{Itype}\mathfrak{L}_{ab}(G)$, where $x \in G - \{1\}$.
Therefore $|\mathbf{Itype}\mathfrak{L}_{ab}(G)| \geqslant 2$.

But **Tarski monsters** T have $|\mathbf{Itype}\mathfrak{L}_{ab}(T)| = 2$.

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Let G be a locally soluble group. Then

$|\mathbf{Itype}\mathfrak{L}_{ab}(G)| = 2 \Leftrightarrow$ either $|G|$ a prime or G infinite cyclic.

First remarks - $|\mathbf{Itype}\mathfrak{L}(G)|$, $|\mathbf{Itype}\mathfrak{L}_{ab}(G)|$

Proposition

$|\mathbf{Itype}\mathfrak{L}_{ab}(G) = 2 \Leftrightarrow$ either $|G|$ a prime or G infinite cyclic.

Proof. If G is cyclic, either infinite or of prime order, then obviously $\mathbf{Itype}\mathfrak{L}(G) = \{\{1\}, G\}$.

Conversely, assume $|\mathbf{Itype}\mathfrak{L}_{ab}(G)| = 2$. We show that G is abelian. Can suppose G finitely generated. Let A be a maximal normal subgroup of G. Then either $|A| = p$, p a prime or A is infinite cyclic. Moreover $B \simeq A$ for every non-trivial abelian subgroup of G. Then it is easy to prove that $C_G(A) = A$. If $|A| = p$, then from $|G/C_G(A)| \leq p - 1$ we get $G = C_G(A) = A$. If $A = \langle a \rangle$ is infinite cyclic, then $a^x = x^{-1}$ for any $x \notin C_G(A)$, but $x^2 \in A$ implies $x^2 = x^{-2}$ a contradiction. Thus again $G = C_G(A) = A$, as required. //

First remarks - $|\mathbf{Itype}\mathfrak{L}(G)|$, $|\mathbf{Itype}\mathfrak{L}_{ab}(G)|$

Problem

What about groups with $\mathbf{Itype}\mathfrak{L}_{ab}(G)$ finite?

Example

If G is a **finitely generated abelian group,** then $\mathbf{Itype}\mathfrak{L}_{ab}(G) = \mathbf{Itype}\mathfrak{L}(G)$ is finite.

First remarks - $|\text{Itype}\mathfrak{L}(G)|$, $|\text{Itype}\mathfrak{L}_{ab}(G)|$

Remark

Using a result due to V.S. Charin it follows that if a group G is such that **Itype**$\mathfrak{L}(G)$ or **Itype**$\mathfrak{L}_{ab}(G)$ is **finite**, then every abelian subgroup of G is minimax.

Definition

A group G is said to be **minimax** if it has a finite series whose factors satisfy $\mathcal{M}in$ or $\mathcal{M}ax$.

V.S. Charin, On soluble groups of type A_4, *Mat. Sbornik* **52** (1960), no. 3, 895-914.

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L.A. Kurdachenko, P. Longobardi, M. M., I.Ya Subbotin

Groups with finitely many

isomorphic classes of non-abelian subgroups

submitted.

Itype$\mathfrak{L}_{non-ab}(G)$ finite

Problem

What about $|\text{Itype}\mathfrak{L}_{non-ab}(G)|$?

Examples

If G is an abelian groups or a minimal non-abelian group , then **Itype$\mathfrak{L}_{non-ab}(G)$** is **finite**.

First remarks - $|\mathbf{Itype}\mathfrak{L}_{non-ab}(G)|$

Groups G with $|\mathbf{Itype}\mathfrak{L}_{non-ab}(G)| = 1$

have been studied by **H. Smith** and **J. Wiegold** in 1997.

Among other results they proved:

Theorem

Let G be a soluble group.

If G is isomorphic to every non abelian subgroup,

then G contains an abelian normal subgroup of prime index.

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Examples - Itype$\mathfrak{L}_{non-ab}(G)$ finite

Remark

If G is a finitely generated abelian-by-finite group,

then

Itype$\mathfrak{L}_{non-ab}(G)$ is **finite**.

Proof.
There exists a normal abelian subgroup A of G with $|G/A| = n$.
Then A is finitely generated, say m-generated.
Every subgroup H of G is an extension of the abelian group $H \cap A$ generated by $\leq m$ elements with a finite group of order $\leq n$. //

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Lemma 1

Let G be a group with **Itype**$\mathfrak{L}_{non-ab}(G)$ **finite**.
If K is an infinite locally finite subgroup of G,
then K is abelian.

Proof. Suppose that K is non-abelian. Being locally finite, K includes a finite non-abelian subgroup F. Then G has an ascending chain

$$F = F_0 \leq F_1 \leq \cdots \leq F_n \leq F_{n+1} \leq \ldots$$

of finite subgroups such that $|F_n| < |F_{n+1}|$ for each $n \in \mathbb{N}$. But in this case, the subgroups F_n and F_m cannot be isomorphic for $n, m \in \mathbb{N}$, $n \neq m$, and we obtain a contradiction. //

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New results - Itype$\mathfrak{L}_{non-ab}(G)$ finite

Definition

A group G is called **generalized radical** if

G has an ascending series

whose factors are either locally nilpotent or locally finite.

Definition

A group G is called **generalized coradical** if

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New results - **Itype**$\mathfrak{L}_{non-ab}(G)$ **finite**

Theorem A

Let G be a non-abelian locally generalized radical group.
If **Itype**$\mathfrak{L}_{non-ab}(G)$ is **finite**,
then G is a minimax, abelian-by-finite group,
with **Tor**(G) finite.

Definition

Tor(G) is the maximal normal torsion subgroup of G.

New results - **Itype**$\mathfrak{L}_{non-ab}(G)$ **finite**

Theorem B

Let G be a non-abelian generalized coradical group.

If **Itype**$\mathfrak{L}_{non-ab}(G)$ is **finite**,

then G is a minimax, abelian-by-finite group with **Tor**(G) finite.

Remark

The **converse** of Theorem A and the converse of Theorem B do not hold.

Theorem B

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then G is a minimax, abelian-by-finite group with **Tor**(G) finite.

Remark

The **converse** of Theorem A and the converse of Theorem B do not hold.

Corollary

Let G be a non-abelian finitely generated generalized radical or coradical group.

Itype$\mathfrak{L}_{non-ab}(G)$ is **finite**,
if and only if G is abelian-by-finite.

Corollary

Let G be a finitely generated generalized radical or coradical group.

Itype$\mathfrak{L}_{non-ab}(G)$ is **finite**,
if and only if
either G is abelian or **Itype**$\mathfrak{L}(G)$ is **finite** .

Corollary

Let G be a non-abelian finitely generated generalized radical or coradical group.

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Corollary

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Problems

Problem

Find a characterization of

abelian-by-finite minimax groups G

with

Itype$\mathfrak{L}_{non-ab}(G)$ **finite.**

Problem

Is there a non-abelian group G in which

Itype$\mathfrak{L}_{non-ab}(G)$ is **finite**

but

Itype$\mathfrak{L}(G)$ is **infinite?**

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Problems

Let G be a **group** and let $\mathfrak{M}$ be a **family of subgroups** of G.

Definitions

Consider the equivalence relation in $\mathfrak{M}$ given by
$H \simeq K$, with $H, K \in \mathfrak{M}$.

Call the isomorphic type **Itype**$\mathfrak{M}$ of $\mathfrak{M}$
any set of representatives of all equivalence classes in $\mathfrak{M}$.

We study groups G in which

Itype$\mathfrak{M}$ is **finite**.

Problems - **Itype$\mathfrak{M}$ finite**

Let G be a group, and let $\mathfrak{M}$ be the family of the commutator subgroups of all subgroups of G :

$$\mathfrak{M} = \{H' \mid H \in \mathfrak{L}(G)\}.$$

The problem to study the structure of the group G in which **Itype$\mathfrak{M}$** is **finite** has been studied by **F. de Giovanni** and **D.J.S. Robinson** in 2005,

as well as by **M. Herzog**, **P. Longobardi**, **M. M.** and **D.J.S. Robinson**, **H. Smith** in a series of papers (2006, 2013, 2014).

Problems - |Itype$\mathfrak{M}$|

F. de Giovanni, D.J.S. Robinson, Groups with finitely many derived subgroups, *J. London Math. Soc.* **71** (2005), no. 2, 658-668.

M. Herzog, P. Longobardi, M. M., On the number of commutators in groups, *Ischia Group Theory 2004, Contemp. Math. Amer. Math. Soc., Providence, RI* **402** (2006), 181-192.

P. Longobardi, M. M., D.J.S. Robinson, H. Smith, On groups with two isomorphism classes of derived subgroups, *Glasgow Math. J.* **55** (2013), no. 3, 655-668.

P. Longobardi, M. M., D.J.S. Robinson, Recent results on groups with few isomorphism classes of derived subgroups, *Proc. of "Group Theory, Combinatorics, and Computing", Boca Raton-Florida, Contemp. Math.* **611** (2014), 121-135.

P. Longobardi, M. M., D.J.S. Robinson, Locally finite groups with finitely many isomorphism classes of derived subgroups, *J. Algebra*, **393** (2013), 102-119.

Now together with **L.A. Kurdachenko** and **P. Longobardi**,

we are considering

groups in which the family $\mathfrak{M}$ of all **non-normal**

is **finite**.

S.N. Černikov, Infinite groups with prescribed properties of their systems of infinite subgroups, *Dokl. Akad. Nauk SSSR*, **159** (1964) 759-760 (Russian), *Soviet Math. Dokl.*, **5** (1964), 1610-1611.

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S.N. Černikov, Infinite nonabelian groups with the minimal condition for noninvariant abelian subgroups, *Dokl. Akad. Nauk SSSR*, **184** (1969) 786-789 (Russian), *Soviet Math. Dokl.*, **10** (1969), 172-175.

G. Cutolo, On groups satisfying the maximal condition on nonnormal subgroups, *Riv. Mat. Pura Appl.*, **9** (1991), 49-59.

R.E. Phillips, J.S. Wilson, On certain minimal conditions for infinite groups, *J. Algebra* **51** (1951), 41-68.

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Thank you for the attention !