On a finiteness condition on non-abelian subgroups

Mercede MAJ

UNIVERSITÀ DEGLI STUDI DI SALERNO

Groups St Andrews 2017 in Birmingham

University of Birmingham

Birmingham

August 5 - 13, 2017
Let $G$ be a group and let $\mathcal{M}$ be a family of subgroups of $G$.

**Basic Problem**

**Main Problem**

Obtain information about the structure of $G$ by looking at properties concerning $\mathcal{M}$. 

Mercede Maj - University of Salerno

On a finiteness condition on non-abelian subgroups
Basic Problem

Let $G$ be a (possibly infinite) group and let $\mathcal{M}$ be a family of subgroups of $G$.

Main Problem

*Find information about the structure of $G$ assuming that $\mathcal{M}$ satisfies a finiteness condition.*
Let $G$ be a (possibly infinite) group.

**Example**

Let $\mathcal{M} = \mathcal{L}(G)$ be the family of all subgroups of $G$. Then

$$\mathcal{L}(G) \text{ is finite } \iff G \text{ is finite}.$$
Let $G$ be a (possibly infinite) group.

**Example**

Let $\mathcal{M} = \mathcal{L}(G)$ be the family of all subgroups of $G$. Then

\[
\mathcal{L}(G) \text{ is finite } \iff G \text{ is finite.}
\]

There are many well-known classical results about classes of groups $G$ with $\mathcal{L}(G) \in \text{Max}$ or $\mathcal{L}(G) \in \text{Min}$. 
Background - $\mathcal{L}(G)$

**Theorem**

Let $G$ be a soluble group. $\mathcal{L}(G)$ has Max if and only if $G$ is polycyclic.

**Definition**

A group $G$ is said to be polycyclic if it has a finite series whose factors are cyclic.
Let $G$ be a soluble group. 
\[ \mathcal{L}(G) \] has \textbf{Max} if and only if $G$ is polycyclic.

\textbf{Definition}

A group $G$ is said to be \textbf{polycyclic} if it has a \textit{finite series} whose factors are \textit{cyclic}.
Let $G$ be a soluble group. $\mathfrak{L}(G)$ has $Min$ if and only if $G$ has an abelian subgroup $A$ of finite index such that $A$ is direct product of finitely many quasi-cyclic groups.

S.N. Černikov, Infinite locally soluble groups, *Mat. Sb.*, 7 (1940) 35-64.
Theorem, (S.N. Černikov)

Let $G$ be a soluble group. \( \mathcal{L}(G) \) has \textit{Min} if and only if

$G$ has an abelian subgroup $A$ of finite index such that

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S.N. Černikov, Infinite locally soluble groups, \textit{Mat. Sb.}, 7 (1940) 35-64.
Let $\mathcal{P}$ be a group theoretical property.

Denote by $\mathcal{L}_\mathcal{P}(G)$ the family of all subgroups $H$ of $G$ such that $H$ has $\mathcal{P}$ and by $\mathcal{L}_{non-\mathcal{P}}(G)$ the family of subgroups $H$ of $G$ such that $H$ does not have $\mathcal{P}$. 
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Denote by $\mathfrak{L}_\mathcal{P}(G)$

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and by $\mathfrak{L}_{\text{non-}\mathcal{P}}(G)$

the family of subgroups $H$ of $G$ such that $H$ does not have $\mathcal{P}$.
Background - $\mathcal{L}_P(G)$, $\mathcal{L}_{\text{non-}P}(G)$

Example

If $\mathcal{L}_{\text{non-}P}(G) = \{G\}$,
then every proper subgroup of $G$ has $P$.

Groups $G$ with finiteness conditions
on $\mathcal{L}_P(G)$ or on $\mathcal{L}_{\text{non-}P}(G)$
for various properties $P$ have been studied by many authors.
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If $\mathcal{L}_{\text{non-}P}(G) = \{G\}$, then every proper subgroup of $G$ has $P$.

Groups $G$ with finiteness conditions on $\mathcal{L}_P(G)$ or on $\mathcal{L}_{\text{non-}P}(G)$ for various properties $P$ have been studied by many authors.
Example

If $\mathcal{P} = ab$ is the property to be abelian, then $\mathcal{L}_{ab}(G)$ is finite $\iff G$ is finite.

Remark

Abelian and Minimal non-abelian groups are groups $G$ with $\mathcal{L}_{non-ab}(G)$ finite.
Let $\mathcal{P} = ab$ be the property to be abelian.

Groups $G$ in which $\mathcal{L}_{ab}(G)$, ordered by inclusion, has $\text{Max}$ or $\text{Min}$ have been firstly studied respectively by A.I. Mal’cev in 1956 and O.J. Schmidt in 1945.

Let \( \mathcal{P} = ab \) be the property to be abelian.

Groups \( G \) in which \( \mathcal{L}_{ab}(G) \), ordered by inclusion, has Max or Min have been firstly studied respectively by A.I. Mal’cev in 1956 and O.J. Schmidt in 1945.


Background - $\mathcal{L}_{\text{non-ab}}(G)$

Groups $G$ in which $\mathcal{L}_{\text{non-ab}}(G)$ has $\text{Max}$ have been studied by


Groups $G$ in which $\mathcal{L}_{\text{non-ab}}(G)$ has Max have been studied by L.A. Kurdachenko and D.I. Zaicev in 1991.

Groups in which $\mathcal{L}_{non\text{-}ab}(G)$ has $Min$

have been studied by


Groups in which $\mathcal{L}_{\text{non-ab}}(G)$ has $\text{Min}$ have been studied by S.N. Černikov in 1964 and 1967.


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Some rather exotic examples of groups can be found in studying this type of problems.

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Remark

Tarski monsters are groups in which

\[ \mathcal{L}(G) \text{ has Max, Min,} \]

\[ \mathcal{L}_{ab}(G) \text{ has Max, Min,} \]

\[ \mathcal{L}_{non-ab}(G) \text{ is finite.} \]
Theorem, (B.I. Plotkin, 1956)

Let $G$ be a radical group.

$\mathcal{L}_{ab}(G)$ has $Min$, if and only if $G$ is soluble and $\mathcal{L}(G)$ has $Min$.

$\mathcal{L}_{non-ab}(G)$ has $Min$, if and only if either $G$ is abelian or $G$ is soluble and $\mathcal{L}(G)$ has $Min$.

Definition

A group $G$ is called radical if there exists an ascending series of $G$ with locally nilpotent factors.
Some sample results

Theorem, (B.I. Plotkin, 1956)

Let $G$ be a radical group.

\[ \mathcal{L}_{ab}(G) \text{ has } \mathcal{M}in, \]
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either $G$ is abelian or $G$ is soluble and $\mathcal{L}(G)$ has $\mathcal{M}in$.

Definition

A group $G$ is called radical if there exists
an ascending series of $G$ with locally nilpotent factors.
L.A. Kurdachenko, P. Longobardi, M. M., I.Ya Subbotin

Groups with finitely many isomorphic classes of non-abelian subgroups

submitted.
A new problem

We study

another finiteness condition on

$\mathcal{L}_P(G)$ and $\mathcal{L}_{\text{non-P}}(G)$. 
Let $G$ be a group and let $M$ be a family of subgroups of $G$.

**Definition**

Consider the equivalence relation in $M$ given by $H \cong K$, with $H, K \in M$.

Call isomorphic type $\text{Itype}_M$ of $M$ any set of representatives of all equivalence classes in $M$. 
Let $G$ be a **group** and let $\mathcal{M}$ be a **family of subgroups** of $G$.

**Definitions**

Consider the equivalence relation in $\mathcal{M}$ given by $H \cong K$, with $H, K \in \mathcal{M}$.

Call the **isomorphic type** $\text{Itype}_\mathcal{M}$ of $\mathcal{M}$ any set of **representatives** of all equivalence classes in $\mathcal{M}$.

We study groups $G$ in which $\text{Itype}_\mathcal{M}$ is finite.
First remarks - $|Itype\mathcal{L}(G)|$

Let $G$ be a group.

**Remark**

If $G$ is non-trivial, then $G, \{1\} \in Itype\mathcal{L}(G)$. Thus

$$|Itype\mathcal{L}(G)| \geq 2.$$ 

**Proposition**

$$|Itype\mathcal{L}(G)| = 2 \iff \text{either } |G| \text{ a prime or } G \text{ infinite cyclic.}$$
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First remarks - $|\text{ltype}\mathcal{L}_{ab}(G)|$

Problem

What about $|\text{ltype}\mathcal{L}_{ab}(G)|$?

Remark

Obviously, if $G \neq \{1\}$, then $\{1\}, <x> \in \text{ltype}\mathcal{L}_{ab}(G)$, where $x \in G - \{1\}$. Therefore $|\text{ltype}\mathcal{L}_{ab}(G)| \geq 2$.

But Tarski monsters $T$ have $|\text{ltype}\mathcal{L}_{ab}(T)| = 2$. 
First remarks - $|\text{Itype} \mathcal{L}_{ab}(G)|$

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Tarski monsters $T$ have $\vert \text{Itype}L_{ab}(T) \vert = 2$.

Proposition

Let $G$ be a locally soluble group. Then

$\vert \text{Itype}L_{ab}(G) \vert = 2 \iff$ either $\vert G \vert$ a prime or $G$ infinite cyclic.
Proposition

\[ |\text{Itype} \mathcal{L}_{ab}(G) | = 2 \iff \text{either } |G| \text{ a prime or } G \text{ infinite cyclic}. \]

Proof. If \( G \) is cyclic, either infinite or of prime order, then obviously \( |\text{Itype} \mathcal{L}(G) | = \{\{1\}, G\} \).

Conversely, assume \( |\text{Itype} \mathcal{L}_{ab}(G) | = 2 \). We show that \( G \) is abelian. Can suppose \( G \) finitely generated. Let \( A \) be a maximal normal subgroup of \( G \). Then either \( |A| = p \), \( p \) a prime or \( A \) is infinite cyclic. Moreover \( B \cong A \) for every non-trivial abelian subgroup of \( G \). Then it is easy to prove that \( C_G(A) = A \). If \( |A| = p \), then from \( |G/C_G(A)| \leq p - 1 \) we get \( G = C_G(A) = A \). If \( A = \langle a \rangle \) is infinite cyclic, then \( a^x = x^{-1} \) for any \( x \notin C_G(A) \), but \( x^2 \in A \) implies \( x^2 = x^{-2} \) a contradiction. Thus again \( G = C_G(A) = A \), as required. //
### Problem

What about groups with $\text{Itype} \mathcal{L}_{ab}(G)$ finite?

### Example

If $G$ is a finitely generated abelian group, then $\text{Itype} \mathcal{L}_{ab}(G) = \text{Itype} \mathcal{L}(G)$ is finite.
Remark

Using a result due to V.S. Charin it follows that if a group $G$ is such that $\text{Itype}_L(G)$ or $\text{Itype}_{ab}(G)$ is finite, then every abelian subgroup of $G$ is minimax.

Definition

A group $G$ is said to be minimax if it has a finite series whose factors satisfy $\text{Min}$ or $\text{Max}$.

First remarks - $|\text{Itype}_L(G)|$, $|\text{Itype}_L^{ab}(G)|$

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Groups with finitely many isomorphistic classes of non-abelian subgroups

submitted.
Problem

What about $|\text{Itype} L_{\text{non-\,ab}}(G)|$?

Examples

If $G$ is an abelian groups or a minimal non-abelian group , then $\text{Itype} L_{\text{non-\,ab}}(G)$ is finite.
Groups $G$ with $|\text{Itype}_{\text{non-}ab}(G)| = 1$

have been studied by H. Smith and J. Wiegold in 1997.

Among other results they proved:

**Theorem**

Let $G$ be a soluble group.

If $G$ is isomorphic to every non abelian subgroup, then $G$ contains an abelian normal subgroup of prime index.

Groups $G$ with $|\text{Itype}\mathfrak{L}_{\text{non-ab}}(G)| = 1$

have been studied by H. Smith and J. Wiegold in 1997.

Among other results they proved:

**Theorem**

Let $G$ be a *soluble* group.

If $G$ is isomorphic to every non abelian subgroup, then $G$ contains an *abelian normal* subgroup of prime index.

Remark

If $G$ is a finitely generated abelian-by-finite group, then

$$\text{Itype}_L^{\text{non-}ab}(G)$$

is finite.

Proof.
There exists a normal abelian subgroup $A$ of $G$ with $|G/A| = n$. Then $A$ is finitely generated, say $m$-generated. Every subgroup $H$ of $G$ is an extension of the abelian group $H \cap A$ generated by $\leq m$ elements with a finite group of order $\leq n$. ◇
Remark

If \( G \) is a finitely generated abelian-by-finite group,
then
\[
\text{Itype} \mathcal{L}_{non-ab}(G) \text{ is finite.}
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Proof.
There exists a normal abelian subgroup \( A \) of \( G \) with \( |G/A| = n \).
Then \( A \) is finitely generated, say \( m \)-generated.
Every subgroup \( H \) of \( G \) is an extension of the abelian group \( H \cap A \)
generated by \( \leq m \) elements with a finite group of order \( \leq n \).  //
Lemma 1

Let $G$ be a group with $\text{Itype}_\text{non-}ab(G)$ finite. If $K$ is an infinite locally finite subgroup of $G$, then $K$ is abelian.

Proof. Suppose that $K$ is non-abelian. Being locally finite, $K$ includes a finite non-abelian subgroup $F$. Then $G$ has an ascending chain

$$F = F_0 \leq F_1 \leq \cdots \leq F_n \leq F_{n+1} \leq \cdots$$

of finite subgroups such that $|F_n| < |F_{n+1}|$ for each $n \in \mathbb{N}$. But in this case, the subgroups $F_n$ and $F_m$ cannot be isomorphic for $n, m \in \mathbb{N}$, $n \neq m$, and we obtain a contradiction. //
Lemma 1

Let $G$ be a group with $\text{Itype}_\text{non-ab}(G)$ finite. If $K$ is an infinite locally finite subgroup of $G$, then $K$ is abelian.

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Definition

A group $G$ is called **generalized radical** if

$G$ has an **ascending** series

whose factors are either **locally nilpotent** or **locally finite**.

Definition

A group $G$ is called **generalized coradical** if

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- $G$ has an **descending** series
- whose factors are either **locally nilpotent** or **locally finite**.
New results - $\text{Itype}L_{non-ab}(G)$ finite

Theorem A

Let $G$ be a non-abelian locally generalized radical group. If $\text{Itype}L_{non-ab}(G)$ is finite, then $G$ is a minimax, abelian-by-finite group, with $\text{Tor}(G)$ finite.

Definition

$\text{Tor}(G)$ is the maximal normal torsion subgroup of $G$. 
New results - $\text{Itype}\mathcal{L}_{non-ab}(G)$ finite

**Theorem B**

Let $G$ be a non-abelian generalized coradical group. If $\text{Itype}\mathcal{L}_{non-ab}(G)$ is finite, then $G$ is a minimax, abelian-by-finite group with $\text{Tor}(G)$ finite.

**Remark**

The converse of Theorem A and the converse of Theorem B do not hold.
Theorem B

Let $G$ be a non-abelian generalized coradical group.

If $\text{Itype}_\text{non-ab}(G)$ is finite,
then $G$ is a minimax, abelian-by-finite group
with $\text{Tor}(G)$ finite.

Remark

The converse of Theorem A and
the converse of Theorem B do not hold.
New results - \( \mathfrak{L}_{non-ab}(G) \) finite

Corollary

Let \( G \) be a non-abelian \textit{finitely generated} generalized radical or coradical group.

\[ \mathfrak{L}_{non-ab}(G) \text{ is finite, if and only if } G \text{ is abelian-by-finite.} \]

Corollary

Let \( G \) be a \textit{finitely generated} generalized radical or coradical group.

\[ \mathfrak{L}_{non-ab}(G) \text{ is finite, if and only if } \text{either } G \text{ is abelian or } \mathfrak{L}(G) \text{ is finite.} \]
New results - $\text{Itype}_{\text{non-\hspace{1pt}ab}}(G)$ finite

**Corollary**

Let $G$ be a non-abelian finitely generated generalized radical or coradical group.

$\text{Itype}_{\text{non-\hspace{1pt}ab}}(G)$ is finite, if and only if $G$ is abelian-by-finite.

**Corollary**

Let $G$ be a finitely generated generalized radical or coradical group.

$\text{Itype}_{\text{non-\hspace{1pt}ab}}(G)$ is finite, if and only if either $G$ is abelian or $\text{Itype}(G)$ is finite.
Problems

Problem

Find a characterization of

\textbf{abelian-by-finite minimax groups} \( G \)

with

\[ \text{Itype}_{\text{non–ab}}(G) \text{ finite.} \]

Problem

Is there a non-abelian group \( G \) in which

\[ \text{Itype}_{\text{non–ab}}(G) \text{ is finite} \]

but

\[ \text{Itype}(G) \text{ is infinite?} \]
Problems

Problem

Find a characterization of

abelian-by-finite minimax groups $G$

with

$l_{\text{type}L_{\text{non-}ab}}(G)$ finite.

Problem

Is there a non-abelian group $G$ in which

$l_{\text{type}L_{\text{non-}ab}}(G)$ is finite

but

$l_{\text{type}L}(G)$ is infinite?
Let $G$ be a group and let $\mathcal{M}$ be a family of subgroups of $G$.

**Definitions**

Consider the equivalence relation in $\mathcal{M}$ given by $H \cong K$, with $H, K \in \mathcal{M}$.

Call the isomorphic type $\text{Itype}_\mathcal{M}$ of $\mathcal{M}$ any set of representatives of all equivalence classes in $\mathcal{M}$.

We study groups $G$ in which $\text{Itype}_\mathcal{M}$ is finite.
Let $G$ be a group, and let $\mathcal{M}$ be the family of the \textit{commutator subgroups} of all subgroups of $G$:

\[ \mathcal{M} = \{ H' \mid H \in \mathcal{L}(G) \} . \]

The problem to study the structure of the group $G$ in which \textit{ItypeM} is \textit{finite} has been studied by F. de Giovanni and D.J.S. Robinson in 2005,


Now together with L.A. Kurdachenko and P. Longobardi, we are considering groups in which the family $\mathcal{M}$ of all non-normal is finite.
\[ L_\mathbb{m} \text{ and } L_{\text{non-}\mathbb{m}}, \quad \mathbb{m} = \text{non-normal subgroups of } G \]

R.E. Phillips, J.S. Wilson, On certain minimal conditions for infinite groups, *J. Algebra* 51 (1951), 41-68.
M. Maj  
Dipartimento di Matematica  
Università di Salerno  
via Giovanni Paolo II, 132, 84084 Fisciano (Salerno), Italy  
E-mail address : mmaj@unisa.it


Thank you for the attention!