Bivariate zeta functions associated to unipotent group schemes

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1 Definitions and motivation

2 Properties

3 Examples
Let $G$ be a group and, for each $n \in \mathbb{N}$, set

\begin{align*}
r_n(G) &= \text{number of } n\text{-dimensional irreducible complex representations of } G \text{ up to isomorphism}, \\
c_n(G) &= \text{number of conjugacy classes of } G \text{ of size } n.
\end{align*}
If these numbers are all finite, we define the *representation* and *conjugacy class zeta functions* of $G$ as

$$\zeta_{irr}^G(s) = \sum_{n=1}^{\infty} \frac{r_n(G)}{n^s},$$

$$\zeta_{cc}^G(s) = \sum_{n=1}^{\infty} \frac{c_n(G)}{n^s},$$

where $s$ is a complex variable.
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where $s$ is a complex variable.

These generating functions converge on some open complex half-plane if the sequences $(r_n(G))$ and $(c_n(G))$ grow at most polynomially.
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If $\Lambda$ has nilpotency class $c$ satisfying $\Lambda' \subset c!\Lambda$, one may define a unipotent group scheme $G_{\Lambda}$ from $\Lambda$ by setting for each $\mathcal{O}$-module $R$,

$$\Lambda(R) := \Lambda \otimes R,$$

and then defining the group operation of $\Lambda(R)$ by means of the Hausdorff series.
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For instance, if $R$ is the completion $O_p$ of $O$ at the nonzero prime ideal $p$. 
Given $r, c \in \mathbb{Z}_{>0}$, consider the free $\mathbb{Z}$-Lie lattice $f_{r,c}$ of nilpotency class $c$ on $r$ generators. We denote by $F_{r,c}$ the group scheme associated to $f_{r,c}$.
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In particular, $f_{2,2} = \langle x, y, z \mid [x, y] = z \rangle$, and $F_{2,2}(\mathbb{Z})$ is the Heisenberg group of the upper unitriangular matrices of $\text{SL}_3(\mathbb{Z})$. 
The bivariate representation and conjugacy class zeta functions of the group $G_\Lambda(\mathcal{O})$ are, respectively,

$$Z_{G_\Lambda(\mathcal{O})}^{\text{irr}}(s_1, s_2) = \sum_{\{0\} \neq I \leq \mathcal{O}} \zeta_{G_\Lambda(\mathcal{O}/I)}^{\text{irr}}(s_1)|\mathcal{O} : I|^{-s_2},$$

and

$$Z_{G_\Lambda(\mathcal{O})}^{\text{cc}}(s_1, s_2) = \sum_{\{0\} \neq I \leq \mathcal{O}} \zeta_{G_\Lambda(\mathcal{O}/I)}^{\text{cc}}(s_1)|\mathcal{O} : I|^{-s_2},$$

where $s_1, s_2$ are complex variables.
Univariate specializations

Given a finite group $G$, the number $\zeta_{G}^{\text{irr}}(0) = \zeta_{G}^{\text{cc}}(0)$ correspond to the total number of irreducible complex representations of $G$, equivalently, the total number of conjugacy classes of $G$. 
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Notice that

$$\zeta^{\text{irr}}_{G_\Lambda(O)}(0, s) = \zeta^{\text{cc}}_{G_\Lambda(O)}(0, s) = \sum_{\{0\} \neq I \leq O} k(G_\Lambda(O/I)) |O : I|^{-s}$$

$$=: \zeta^{k}_{G_\Lambda(O)}(s),$$

is the \textit{class number zeta function}.
Moreover, if $G_\Lambda(\mathcal{O})$ has nilpotency class 2, there exists also a univariate specialization of $\mathcal{Z}^{\text{irr}}_{G_\Lambda(\mathcal{O})}(s_1, s_2)$ which gives the (twist-isoclass) zeta function of $G_\Lambda(\mathcal{O})$. 
These generating functions satisfy the following Euler decomposition

\[ \mathcal{Z}^*_{G_\Lambda(O)}(s_1, s_2) = \prod_{p \in \text{Spec}(O)} \mathcal{Z}^*_{G_\Lambda(O_p)}(s_1, s_2), \]

where \( * \in \{\text{irr, cc}\} \)
These generating functions satisfy the following Euler decomposition

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where \( * \in \{ \text{irr, cc} \} \) and the local factors are given by

\[ Z^*_{G^\Lambda}(O_p)(s_1, s_2) = \sum_{i=0}^{\infty} \zeta^*_{G^\Lambda}(O_p/p^i)(s_1) |O_p : p|^{-is_2}, \]

where \( O_p \) denotes the completion of \( O \) at the nonzero prime ideal \( p \).
Theorem (L.)

For each \( * \in \{ \text{irr}, \text{cc} \} \), there exist a positive integer \( t \) and a rational function \( R^*(X_1, \ldots, X_t, Y_1, Y_2) \) in \( \mathbb{Q}(X_1, \ldots, X_t, Y_1, Y_2) \) such that, for all but finitely many nonzero prime ideals \( \mathfrak{p} \) of \( \mathcal{O} \), there exist algebraic integers \( \lambda_1(\mathfrak{p}), \ldots, \lambda_t(\mathfrak{p}) \) for which the following holds.

\[
Z^*_G(\mathcal{O}_p)(s_1, s_2) = R^*(\lambda_1(\mathfrak{p}), \ldots, \lambda_t(\mathfrak{p}), q^{-s_1}, q^{-s_2}),
\]

Moreover, these local factors satisfy the following functional equation

\[
Z^*_G(\mathcal{O}_p)(s_1, s_2) \mid q \rightarrow q^{-1} = -q^{h-s_2} Z^*_G(\mathcal{O}_p)(s_1, s_2),
\]

where \( h = \dim_K(\Lambda \otimes K) \).
Example

Let $\mathfrak{p}$ be a nonzero prime ideal of $\mathcal{O}$ with $q = |\mathcal{O} : \mathfrak{p}|$, and $r, c \in \mathbb{Z}_{>0}$.

The local factor of the conjugacy class zeta function of the group $F_{r,2}(\mathcal{O})$ at $\mathfrak{p}$ is given by

$$Z^{cc}_{F_{r,2}(\mathcal{O})}(\mathfrak{p})(s_1, s_2) = \frac{1 - q^{\binom{r-1}{2} - (r-1)s_1 - s_2}}{(1 - q^{\binom{r}{2} - s_2})(1 - q^{\binom{r}{2} + 1 - (r-1)s_1 - s_2}),$$

where $q = |\mathcal{O} : \mathfrak{p}|$. 
Let $p$ be a nonzero prime ideal of $O$ with $q = |O : p|$, and $r, c \in \mathbb{Z}_{>0}$.

The local factor of the conjugacy class zeta function of the group $F_{r,2}(O)$ at $p$ is given by

$$Z_{cc}^{F_{r,2}(O_p)}(s_1, s_2) = \frac{1 - q^{\binom{r-1}{2} - (r-1)s_1 - s_2}}{(1 - q^{r-2})^{s_2}(1 - q^{r-2} + 1 - (r-1)s_1 - s_2)},$$

where $q = |O : p|$.

This is a rational function and

$$Z_{cc}^{F_{r,2}(O_p)}(s_1, s_2) \big|_{q \to q^{-1}} = -q^{r-2} + r-s_2 \ Z_{F_{r,c}(O_p)}^{cc}(s_1, s_2).$$
The class number zeta function of $F_{r,2}(\mathcal{O}_p)$ is obtained by setting $s_1 = 0$ to

$$Z_{F_{r,2}(\mathcal{O}_p)}^{cc}(s_1, s_2) = \frac{1 - q^{(r-1)- (r-1)s_1-s_2}}{(1 - q^{r-2})^{-s_2}(1 - q^{r-1}+1-(r-1)s_1-s_2)}.$$
The class number zeta function of $F_{r,2}(\mathcal{O}_p)$ is obtained by setting $s_1 = 0$ to

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That is,

$$\zeta^k_{F_{r,2}}(\mathcal{O}_p)(s) = \frac{1 - q^{\binom{r-1}{2} - s}}{(1 - q^{\binom{r}{2} - s})(1 - q^{\binom{r}{2} + 1 - s})}.$$
In particular, for the Heisenberg group of the upper unitriangular matrices of $\text{SL}_3(\mathcal{O})$, we obtain

$$
\mathcal{Z}_{F_2,2}(\mathcal{O}_p)(s_1, s_2) = \frac{1 - q^{-s_1-s_2}}{(1 - q^{1-s_2})(1 - q^{2-s_1-s_2})}.
$$
In particular, for the Heisenberg group of the upper unitriangular matrices of $\text{SL}_3(O)$, we obtain

$$Z_{F_2,2}(O_p)(s_1, s_2) = \frac{1 - q^{-s_1-s_2}}{(1 - q^{1-s_2})(1 - q^{2-s_1-s_2})}.$$ 

Furthermore,

$$Z^\text{irr}_{F_2,2}(O_p)(s_1, s_2) = \frac{1 - q^{-s_1-s_2}}{(1 - q^{1-s_1-s_2})(1 - q^{2-s_2})}.$$
In particular, for the Heisenberg group of the upper unitriangular matrices of $\text{SL}_3(\mathcal{O})$, we obtain

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Furthermore,

$$Z_{F_2,2}^{\text{irr}}(\mathcal{O}_p)(s_1, s_2) = \frac{1 - q^{-s_1-s_2}}{(1 - q^{1-s_1-s_2})(1 - q^{2-s_2})}.$$  

These formulae yield

$$\zeta^k_{F_2,2}(\mathcal{O})(s) = \frac{\zeta_K(s - 2)\zeta_K(s - 1)}{\zeta_K(s)} \zeta_K(s),$$

$$\tilde{\zeta}^{\text{irr}}_{F_2,2}(\mathcal{O})(s) = \frac{\zeta_K(s - 1)}{\zeta_K(s)},$$

where $\zeta_K G(s)$ is the Dedekind zeta function of the group $G$. 
Thank you for your attention!