Bivariate zeta functions associated to unipotent group schemes

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1. Definitions and motivation
2. Properties
3. Examples

Let G be a group and, for each $n \in \mathbb{N}$, set

$$\begin{aligned} r_n(G) = &\text{number of } n\text{-dimensional irreducible complex} \\ &\text{representations of } G \text{ up to isomorphism,} \\ c_n(G) = &\text{number of conjugacy classes of } G \text{ of size } n. \end{aligned}$$

If these numbers are all finite, we define the *representation* and *conjugacy class zeta functions* of G as

$$\zeta_G^{\mathrm{irr}}(s) = \sum_{n=1}^{\infty} \frac{r_n(G)}{n^s},$$
$$\zeta_G^{\mathrm{cc}}(s) = \sum_{n=1}^{\infty} \frac{c_n(G)}{n^s},$$

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These generating functions converge on some open complex half-plane if the sequences $(r_n(G))$ and $(c_n(G))$ grow at most polynomially.

Groups obtained from Lie lattices

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If Λ has nilpotency class c satisfying $\Lambda' \subset c!\Lambda$, one may define a unipotent group scheme $\mathbf{G}_\Lambda$ from Λ by setting for each $\mathcal{O}$-module R,

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For instance, if R is the completion $\mathcal{O}_\mathfrak{p}$ of $\mathcal{O}$ at the nonzero prime ideal $\mathfrak{p}$.

Example

Given $r, c \in \mathbb{Z}_{>0}$, consider the free $\mathbb{Z}$-Lie lattice $\mathfrak{f}_{r,c}$ of nilpotency class c on r generators.
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In particular, $\mathfrak{f}_{2,2} = \langle x, y, z \mid [x, y] = z \rangle$, and $F_{2,2}(\mathbb{Z})$ is the Heisenberg group of the upper unitriangular matrices of $\mathrm{SL}_3(\mathbb{Z})$.

Bivariate zeta functions of groups $\mathbf{G}_\Lambda(\mathcal{O})$

The *bivariate representation* and *conjugacy class zeta functions* of the group $\mathbf{G}_\Lambda(\mathcal{O})$ are, respectively,

$$\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}_\Lambda(\mathcal{O})}(s_1, s_2) = \sum_{\{0\} \neq I \trianglelefteq \mathcal{O}} \zeta^{\mathrm{irr}}_{\mathbf{G}_\Lambda(\mathcal{O}/I)}(s_1) |\mathcal{O} : I|^{-s_2}, \text{ and}$$

$$\mathcal{Z}^{\mathrm{cc}}_{\mathbf{G}_\Lambda(\mathcal{O})}(s_1, s_2) = \sum_{\{0\} \neq I \trianglelefteq \mathcal{O}} \zeta^{\mathrm{cc}}_{\mathbf{G}_\Lambda(\mathcal{O}/I)}(s_1) |\mathcal{O} : I|^{-s_2},$$

where s_1, s_2 are complex variables.

Univariate specializations

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Notice that

$$\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}_\Lambda(\mathcal{O})}(0,s) = \mathcal{Z}^{\mathrm{cc}}_{\mathbf{G}_\Lambda(\mathcal{O})}(0,s) = \sum_{\{0\} \neq I \trianglelefteq \mathcal{O}} \mathrm{k}(\mathbf{G}_\Lambda(\mathcal{O}/I)) |\mathcal{O} : I|^{-s}$$
$$=: \zeta^{\mathrm{k}}_{\mathbf{G}_\Lambda(\mathcal{O})}(s),$$

is the *class number zeta function*.

Moreover, if $\mathbf{G}_\Lambda(\mathcal{O})$ has nilpotency class 2, there exists also a univariate specialization of $\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}_\Lambda(\mathcal{O})}(s_1, s_2)$ which gives the (twist-isoclass) zeta function of $\mathbf{G}_\Lambda(\mathcal{O})$.

These generating functions satisfy the following Euler decomposition

$$\mathcal{Z}^{*}_{\mathbf{G}_{\Lambda}(\mathcal{O})}(s_1, s_2) = \prod_{\mathfrak{p} \in \mathrm{Spec}(\mathcal{O})} \mathcal{Z}^{*}_{\mathbf{G}_{\Lambda}(\mathcal{O}_{\mathfrak{p}})}(s_1, s_2),$$

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where $* \in \{\mathrm{irr}, \mathrm{cc}\}$ and the local factors are given by

$$\mathcal{Z}^*_{\mathbf{G}_\Lambda(\mathcal{O}_\mathfrak{p})}(s_1, s_2) = \sum_{i=0}^{\infty} \zeta^*_{\mathbf{G}_\Lambda(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^i)}(s_1) |\mathcal{O}_\mathfrak{p} : \mathfrak{p}|^{-is_2},$$

where $\mathcal{O}_\mathfrak{p}$ denotes the completion of $\mathcal{O}$ at the nonzero prime ideal $\mathfrak{p}$.

Uniformity and functional equations of the local factors

Theorem (L.)

For each $* \in \{\mathrm{irr}, \mathrm{cc}\}$, *there exist a positive integer* t *and a rational function* $R^*(X_1, \ldots, X_t, Y_1, Y_2)$ *in* $\mathbb{Q}(X_1, \ldots, X_t, Y_1, Y_2)$ *such that, for all but finitely many nonzero prime ideals* $\mathfrak{p}$ *of* $\mathcal{O}$, *there exist algebraic integers* $\lambda_1(\mathfrak{p}), \ldots, \lambda_t(\mathfrak{p})$ *for which the following holds.*

$$\mathcal{Z}^*_{\mathbf{G}(\mathcal{O}_\mathfrak{p})}(s_1, s_2) = R^*(\lambda_1(\mathfrak{p}), \ldots, \lambda_t(\mathfrak{p}), q^{-s_1}, q^{-s_2}),$$

Moreover, these local factors satisfy the following functional equation

$$\mathcal{Z}^*_{\mathbf{G}(\mathcal{O}_\mathfrak{p})}(s_1, s_2) \mid_{\substack{q \to q^{-1} \\ \lambda_j(\mathfrak{p}) \to \lambda_j(\mathfrak{p})^{-1}}} = -q^{h - s_2} \mathcal{Z}^*_{\mathbf{G}(\mathcal{O}_\mathfrak{p})}(s_1, s_2),$$

where $h = \dim_K(\Lambda \otimes K)$.

Example

Let $\mathfrak{p}$ be a nonzero prime ideal of $\mathcal{O}$ with $q = |\mathcal{O} : \mathfrak{p}|$, and $r, c \in \mathbb{Z}_{>0}$.

The local factor of the conjugacy class zeta function of the group $F_{r,2}(\mathcal{O})$ at $\mathfrak{p}$ is given by

$$\mathcal{Z}^{\mathrm{cc}}_{\mathcal{F}_{r,2}(\mathcal{O}_{\mathfrak{p}})}(s_1, s_2) = \frac{1 - q^{\binom{r-1}{2} - (r-1)s_1 - s_2}}{(1 - q^{\binom{r}{2} - s_2})(1 - q^{\binom{r}{2} + 1 - (r-1)s_1 - s_2})},$$

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This is a rational function and

$$\mathcal{Z}^{\mathrm{cc}}_{\mathcal{F}_{r,2}(\mathcal{O}_{\mathfrak{p}})}(s_1, s_2) \mid_{q \to q^{-1}} = -q^{\binom{r}{2} + r - s_2} \mathcal{Z}^{\mathrm{cc}}_{F_{r,c}(\mathcal{O}_{\mathfrak{p}})}(s_1, s_2).$$

The class number zeta function of $F_{r,2}(\mathcal{O}_\mathfrak{p})$ is obtained by setting $s_1 = 0$ to

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That is,

$$\zeta^{\mathsf{k}}_{\mathcal{F}_{r,2}(\mathcal{O}_\mathfrak{p})}(s) = \frac{1 - q^{\binom{r-1}{2} - s}}{(1 - q^{\binom{r}{2} - s})(1 - q^{\binom{r}{2} + 1 - s})}.$$

In particular, for the Heisenberg group of the upper unitriangular matrices of $\mathrm{SL}_3(\mathcal{O})$, we obtain

$$\mathcal{Z}^{\mathrm{cc}}_{F_{2,2}(\mathcal{O}_{\mathfrak{p}})}(s_1, s_2) = \frac{1 - q^{-s_1 - s_2}}{(1 - q^{1-s_2})(1 - q^{2-s_1-s_2})}.$$

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Furthermore,

$$\mathcal{Z}^{\mathrm{irr}}_{F_{2,2}(\mathcal{O}_\mathfrak{p})}(s_1, s_2) = \frac{1 - q^{-s_1-s_2}}{(1 - q^{1-s_1-s_2})(1 - q^{2-s_2})}.$$

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These formulae yield

$$\zeta^{\mathrm{k}}_{F_{2,2}(\mathcal{O})}(s) = \frac{\zeta_K(s-2)\zeta_K(s-1)}{\zeta_K(s)},$$
$$\zeta^{\widetilde{\mathrm{irr}}}_{F_{2,2}(\mathcal{O})}(s) = \frac{\zeta_K(s-1)}{\zeta_K(s)},$$

where $\zeta_{K\,G}(s)$ is the Dedekind zeta function of the group G.

Thank you for your attention!