# A characterization of finite simple groups by the order and graph

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Groups St Andrews, Birmingham, 2017.08.07

# finite groups and quantity

The order of a finite group and the order of its elements are fundamental in the theory of finite groups. Naturally, the question is that what we can say for influence of the order of a finite group and its element orders on structure of a finite group.

# Some early work

W. Burnside question( Quart. J. Pure Appl. Math., 33(1902)): Is a finite group if it is generated by a finite set of elements and the orders of all the elements in the groups are finite?

Let $\pi_e(G)$ denote the set of orders of elements of $G$.

B. H. Neuman(1937): Groups with $\pi_e(G) = \{1, 2, 3\}$.

G. Higman (J. London Math. Soc., 32(1957), 335-342.): If $G$ is solvable and the order of each element in $G$ is a prime number power, then $|G| = p^a q^b$, $a$, $b \geq 0$.

M. Suzuki (Ann. Math., 75(1962), 105-145): All simple groups all of whose elements have prime power order were determined.

Wujie Shi and his cooperator had been studying the characterization of finite simple group by the set of orders of elements and the order of groups since 1981.

In 1987, Shi put forward the following conjecture.

Wujie Shi Conjecture(1987): Let $G$ be a finite group and $T$ be a simple group. Then $G$ is isomorphic to $T$ if and only if $\pi_e(G) = \pi_e(T)$ and $|G| = |T|$.

We obtain the following theorem in 2007.

**Theorem**

*Let $G$ be a finite group and $T$ be a finite simple group. If $|G| = |T|$ and $\pi_e(G) = \pi_e(T)$, then $G \cong T$ or $\{G, T\} = \{B_n(q), C_n(q)\}$, where $n \geq 3$ and $q$ is odd.*

M. A. Grechkoseeva ( Siberian Math. J., 48(1)(2007), 73-75) proved that $\pi_e(B_n(q)) \neq \pi_e(C_n(q))$. Thus the theorem yields a positive answer to Shi's conjecture.

# further questions

The question is which we are interested in what we can say for influence of the subset of $\pi_e(G)$ whose elements are a product of two primes powers on structure of finite groups. In fact, this research has caught the attention of some people.

# Prime graph components of finite groups

Let $G$ be a finite group. Let $\pi(G)$ denote the set of prime divisors of $|G|$. Construct the Gruenberg-Kegel graph of $G$ (also called prime graph) as follows:

Gruenberg-Kegel Graph $\Gamma(G)$ : The set of vertices is $\pi(G)$, two vertices $p$, $q$ are jointed by an edge if and only if $G$ contains an element of order $pq$.

J. S. Williams(Journal of Algebra, 69(1981), 487-513)): Prime graph components of finite simple groups,

# weighted graph

By algorithmic considerations, Kantor and Seress associate a weighted graph $\Delta(X)$ to each Lie type simple group $X$ and proved if $\Delta(G) \cong \Delta(T)$ for Lie type simple groups $G$ and $T$, then $G \cong T$ with an explicit list of exception.

William M. Kantor and Akos Seress, Prime power graphs for groups of Lie Type, Journal of Algebra, 247(2002), 370-434

# definition of weighted graph

For each finite simple group $G$ of Lie type of characteristic $p$, define a graph $\Gamma_1(G)$ as follows: The vertices of $\Gamma_1(G)$ are the prime powers $r^a$ that occur as orders of some elements of $G$, for all primes $r \neq p$ and integers $a > 0$. Prime powers $r^a$, $s^b$ are connected if and only if $G$ has an element of order $lcm(r^a, s^b)$.

Two vertices of $\Gamma_1(G)$ are equivalent if they have the same neighbors, and denote the quotient graph with respect to this equivalence relation by $\Delta(G)$; the vertex set of $\Delta(G)$ is denoted $V(\Delta(G))$. We consider $\Delta(G)$ as a weighted graph: the weight of $v \in V(\Delta(G))$ is the least common multiple of the prime powers in the equivalence class $v$.

# What is our work?

Recently, we consider to characterize finite simple groups by their orders, graph, arithmetic complex and $NC$ prime number.

$NC$ Prime Number : The prime number $r$ in $\pi(G)$ is called an $NC$ prime number of $G$ if for any $s \in \pi(G) \setminus \{r\}$, there exist no edge in prime graph $\Gamma(G)$ which joints $s$ and $r$, that is, $r$ ia a isolated point of the prime graph $\Gamma(G)$.

Prime Power Graph $PP(G)$: The vertices of the prime power graph $PP(G)$ of $G$ are the prime powers in $\pi_e(G)$. Two vertices $p^m$ and $q^n$ are jointed by an edge if and only if $G$ has an element of order $p^m \cdot q^n$.

Simplicial Complex $AC(G)$: The simplicial complex $AC(G)$ whose Complex are those subsets $\{p_1, p_2, \ldots, p_k\}$ of $\pi(G)$ for which $G$ has an element of order $\prod_{i=1}^{k} p_i$ is called the **arithmetic complex** of $G$.

Remark: The spectrum $\pi_e(G)$ determines the prime power graph $PP(G)$ as well as the arithmetic complex $AC(G)$. And $\Gamma(G)$ is a subobject of $PP(G)$ as well as of $AC(G)$.

## The characterization of sporadic simple groups

Firstly, we give the characterization of 26 sporadic simple groups by both the order and $NC$ primes listed in the $NC$ column in the following Table.

**Theorem**

*Let $T$ be one of 26 sporadic simple groups and $G$ a finite group of the same order as $T$. If $NC$ primes of $T$ in the following Table are also $NC$ primes of $G$, then $T \cong G$.*

| $T$ | $NC$ | $T$ | $NC$ | $T$ | $NC$ |
|---|---|---|---|---|---|
| $F_{22}$ | $13$ | $Suz$ | $11, 13$ | $B$ | $31, 47$ |
| $M_{11}$ | $11, 5$ | $HS$ | $7, 11$ | $J_1$ | $7, 11, 19$ |
| $M_{12}$ | $11$ | $J_2$ | $7$ | $He$ | $17$ |
| $Ru$ | $29$ | $O'N$ | $11, 19, 31$ | $M$ | $41, 59, 71$ |
| $Co_3$ | $23$ | $Co_1$ | $23$ | $Co_2$ | $11, 23$ |
| $Th$ | $19, 31$ | $Ly$ | $31, 37, 69$ | $J_4$ | $23, 29, 31, 37, 43$ |
| $McL$ | $11$ | $M_{24}$ | $11, 23$ | $M_{23}$ | $23, 11$ |
| $F_{23}$ | $17, 23$ | $M_{22}$ | $5, 7, 11$ | $F_{24}^{'}$ | $17, 23, 29$ |
| $HN$ | $19$ | $J_3$ | $17, 19$ | | |

# The characterization of alternating groups

Secondly, we prove that alternating groups $A_n$ are determined by both the order and the arithmetic complex $AC(G)$.

**Theorem**

*Let $T$ be an alternating group $A_n$ and $G$ a finite group of the same order as $T$. If $AC(G) = AC(T)$, then $T \cong G$.*

Thirdly, we obtain the characterization of Lie type simple groups by both the order and the prime power graph.

**Theorem**

*Let $T$ be a Lie type simple group and $G$ a finite group of the same order as $T$. If $PP(T) = PP(G)$ and $\{T, G\} \neq \{B_n(q), C_n(q)\}$, where $n \geq 3$ and $q$ is odd, then $T \cong G$.*

Compared with Kantor and Seress's work, we remove the assumption which $G$ is a Lie type simple group. And Shi's conjecture is a corollary of our result and we give a unified proof of Shi's conjecture.

Thank you for your attention!