STABILIZERS OF VERTICES OF GRAPHS WITH PRIMITIVE AUTOMORPHISM GROUPS AND A STRONG VERSION OF THE SIMS CONJECTURE

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Let G be a primitive permutation group on a finite set X and $x \in X$. Let d be the length of some G_x-orbit on $X \setminus \{x\}$. It is easy to see that $d = 1$ implies $G_x = 1$ (and $G \cong \mathbb{Z}_p$ for a prime p) and $d = 2$ implies $G_x \cong \mathbb{Z}_2$ (and $G \cong D_{2p}$ for an odd prime p). In [Math. Z. 95 (1967)], Charles Sims adapted arguments by William Tutte concerning vertex stabilizers of cubic (i.e. of valency 3) graphs in vertex-transitive groups of automorphisms (see [Proc. Camb. Phil. Soc. 43 (1947)] and [Canad. J. Math. 11 (1959)]) to prove that $|d| = 3$ implies $|G_x|$ divides $3 \cdot 2^4$. In connection with this result Sims made the following general conjecture which is now well known as the Sims conjecture.

SIMS CONJECTURE. There exists a function

$$f : \mathbb{N} \longrightarrow \mathbb{N}$$

such that, if G is a primitive permutation group on a finite set X, G_x is the stabilizer in G of a point x from X, and d is the length of some non-trivial G_x-orbit on $X \setminus \{x\}$, then $|G_x| \leq f(d)$.

Some progress toward proving this conjecture had been obtained in papers of Sims (Math. Z. 95 (1967)), Thompson (J. Algebra 14 (1970)), Wielandt (Ohio State Univ. Lecture Notes, 1971), Knapp (Math. Z. 133 (1973), Arch. Math. 36 (1981)), Fomin (In: Sixth All-Union Symp. on Group Theory, Naukova Dumka, Kiev, 1980). In particular, Thompson and independently Wielandt proved that $|G_x/O_p(G_x)|$ is bounded by some function of d for some prime p. But only with the use of the classification of finite simple groups, the validity of the conjecture was proved by Cameron, Praeger, Saxl and Seitz (Bull. London Math. Soc. 15 (1983)).

This proof implies that one can take a function of the form $\exp(Cd^3)$, where C is some constant, as the function $f(d)$ in the Sims conjecture.

The Sims conjecture can be formulated using graphs as follows.

For an undirected connected graph Γ (without loops or multiple edges) with vertex set $V(\Gamma)$, $G \leq Aut(\Gamma)$, $x \in V(\Gamma)$, and $i \in \mathbb{N} \cup \{0\}$, we will denote by $G_x^{[i]}$ the elementwise stabilizer in G of the (closed) ball of radius i of the graph Γ centered at x in the natural metric on $V(\Gamma)$.

Let G be a primitive permutation group on a finite set X and $x, y \in X$, $x \neq y$. Consider the graph $\Gamma_{G,\{x,y\}}$ with vertex set $V(\Gamma_{G,\{x,y\}}) = X$ and edge set $E(\Gamma_{G,\{x,y\}}) = \{\{g(x), g(y)\} | g \in G\}$. Then $\Gamma_{G,\{x,y\}}$ is an undirected connected graph, G is an automorphism group of $\Gamma_{G,\{x,y\}}$ acting primitively on $V(\Gamma_{G,\{x,y\}})$, and the length of the G_x-orbit containing y is equal either to the valency of $\Gamma_{G,\{x,y\}}$ (if there exists an element in G that transposes x and y) or to the half of the valency of $\Gamma_{G,\{x,y\}}$ (otherwise). Now it is easy to see that the Sims conjecture can be reformulated in the following form.

SIMS CONJECTURE (GEOMETRICAL FORM). There exists a function $\psi : \mathbb{N} \cup \{0\} \longrightarrow \mathbb{N}$ such that, if Γ is an undirected connected finite graph and G is its automorphism group acting primitively on $V(\Gamma)$, then $G_x^{[\psi(d)]} = 1$ for $x \in V(\Gamma)$, where d is the valency of the graph Γ.

Using the classification of finite simple groups, the authors obtained in (Dokl. Math. 59 (1999)) the following result, which establishes the validity of a strengthened version of the Sims conjecture.

THEOREM 1. If Γ is an undirected connected finite graph and G its automorphism group acting primitively on $V(\Gamma)$, then $G_x^{[6]} = 1$ for $x \in V(\Gamma)$.

In other words, automorphisms of connected finite graphs with vertex-primitive automorphism groups are determined by images of vertices of any ball of radius 6.

Actually, we proved a result which is stronger than Theorem 1 (Theorem 2 below). It is formulated in terms of subgroup structure of finite groups. To formulate the result, we need the following definitions.

Recall that, for a group G and $H \leq G$, the subgroup $H_G = \bigcap_{g \in G} gHg^{-1}$ is called the core of the subgroup H in G.

For a group G, its subgroups M_1 and M_2, and any $i \in \mathbb{N}$, let us define by induction subgroups $(M_1, M_2)^i$ and $(M_2, M_1)^i$ of $M_1 \cap M_2$, which will be called the ith mutual cores of M_1 with respect to M_2 and of M_2 with respect to M_1, respectively. Put

$$(M_1, M_2)^1 = (M_1 \cap M_2)_{M_1}, \quad (M_2, M_1)^1 = (M_1 \cap M_2)_{M_2}.$$

For $i \in \mathbb{N}$, assuming that $(M_1, M_2)^i$ and $(M_2, M_1)^i$ are already defined, put

$$(M_1, M_2)^{i+1} = ((M_1, M_2)^i \cap (M_2, M_1)^i)_{M_1},$$

$$(M_2, M_1)^{i+1} = ((M_1, M_2)^i \cap (M_2, M_1)^i)_{M_2}.$$

It is clear that

$$(M_1, M_2)^{i+1} = ((M_2, M_1)^i)_{M_1}, \quad (M_2, M_1)^{i+1} = ((M_1, M_2)^i)_{M_2}$$

for all $i \in \mathbb{N}$.

If G is a primitive permutation group on a finite set X and $x, y \in X$, $x \neq y$, then we have the following interpretation of mutual cores $(G_x, G_y)^i$ and $(G_y, G_x)^i$ for $i \in \mathbb{N}$. Let $\Gamma_{G,(x,y)}$ be the directed graph with $V(\Gamma_{G,(x,y)}) = X$ and $E(\Gamma_{G,(x,y)}) = \{(g(x), g(y)) \mid g \in G\}$, i.e. the directed graph corresponding to the orbital of G containing (x, y). Then $(G_x, G_y)^i$ is the pointwise stabilizer in G_x of the set

$$\{z \in V(\Gamma_{G,(x,y)}) \mid \text{there exist } 0 \leq j \leq i \text{ and } z_0, ..., z_j \in V(\Gamma_{G,(x,y)})$$
$$\text{such that } z_0 = x, z_j = z, (z_k, z_{k+1}) \in E(\Gamma_{G,(x,y)}) \text{ for all even}$$
$$0 \leq k < j \text{ and } (z_{k+1}, z_k) \in E(\Gamma_{G,(x,y)}) \text{ for all odd } 0 < k < j\}$$

and $(G_y, G_x)^i$ is the pointwise stabilizer in G_y of the set

$$\{z \in V(\Gamma_{G,(x,y)}) \mid \text{there exist } 0 \leq j \leq i \text{ and } z_0, ..., z_j \in V(\Gamma_{G,(x,y)})$$
$$\text{such that } z_0 = y, z_j = z, (z_{k+1}, z_k) \in E(\Gamma_{G,(x,y)}) \text{ for all even}$$
$$0 \leq k < j \text{ and } (z_k, z_{k+1}) \in E(\Gamma_{G,(x,y)}) \text{ for all odd } 0 < k < j\}.$$

Mutual cores of subgroups M_1 and M_2 of a group G have the following obvious properties. For $i \in \mathbb{N}$, the equality $(M_1, M_2)^i = (M_2, M_1)^i$ means that this subgroup is maximal in $M_1 \cap M_2$, with the property that it is normal both in M_1 and in M_2, and all the groups $(M_1, M_2)^{i+j}$ and $(M_2, M_1)^{i+j}$ for $j \in \mathbb{N}$ coincide with it.

THEOREM 2. **Let G be a finite group, and let M_1 and M_2 be distinct conjugate maximal subgroups of G. Then, the subgroups $(M_1, M_2)^6$ and $(M_2, M_1)^6$ coincide and are normal in the group G.**

Under the hypothesis of Theorem 1 for $|V(\Gamma)| > 1$, if we set $M_1 = G_x$ and $M_2 = G_y$, where x and y are adjacent vertices of the graph Γ, then $G_x^{[i]} \leq (M_1, M_2)^i$ and $G_y^{[i]} \leq (M_2, M_1)^i$ for all $i \in \mathbb{N}$. Thus, Theorem 1 follows from Theorem 2.

The following result is also derived from Theorem 2.

Corollary. *Let G be a finite group, let M_1 be a maximal subgroup of G, and let M_2 be a subgroup of G containing $(M_1)_G$ and not contained in M_1. Then the subgroup $(M_1, M_2)^{12}$ coincides with $(M_1)_G$.*

EXAMPLES

EXAMPLE 1. Let $G = E_8(q)$, where q is a power of a prime p, let M_1 be a parabolic maximal subgroup of G obtained from the Dynkin diagram for E_8 by deleting the root α_4, and let a be an element of the monomial subgroup of G corresponding to the reflection s_{α_4}. Define $Q = O_p(M_1)$ and $M_2 = aM_1a^{-1}$. Let Γ be a graph with the vertex set $\{hM_1h^{-1} \mid h \in G\}$ and the edge set $\{\{hM_1h^{-1}, hM_2h^{-1}\} \mid h \in G\}$. Then, Γ and G satisfy the conditions of Theorem 1. We can show that the series

$$1 = (M_1, M_2)^6 < (M_1, M_2)^5 < (M_1, M_2)^4 <$$
$$(M_1, M_2)^3 < (M_1, M_2)^2 < O_p((M_1, M_2)^1) < Q$$

coincides with the series

$$1 = G_x^{[6]} < G_x^{[5]} < G_x^{[4]} < G_x^{[3]} < G_x^{[2]} < O_p(G_x^{[1]}) < Q,$$

where $x = M_1 \in V(\Gamma)$, as well as with the upper and lower central series of the group Q.

EXAMPLE 2. Take G, M_1, and a as in Example 1. Define

$$M_2 = (M_1 \cap aM_1a^{-1})\langle a\rangle.$$

Then, using the properties from Example 1, it is easy to verify that $(M_1, M_2)^{11} \neq 1$ and $(M_1, M_2)^{12} = 1$.

EXAMPLE 3. For any positive integer n, let A be an elementary abelian 2-group of order 2^{2n+3} with basis $\{a_1, a_2, \ldots, a_{2n+3}\}$, and let t_1, t_2 be involutive automorphisms of A induced by the permutations

$$(a_1 a_2)(a_3 a_4)\ldots(a_{2n+1} a_{2n+2})(a_{2n+3})$$

and

$$(a_1)(a_2 a_3)\ldots(a_{2n+2} a_{2n+3})$$

of this basis, respectively. Define the subgroups $G_n = A\langle t_1,\ t_2\rangle$, $M_{1,n} = \langle a_1, \ldots, a_{2n+2}, t_1\rangle$, and $M_{2,n} = \langle a_2, \ldots, a_{2n+3}, t_2\rangle$ in the holomorph of A. Then, $M_{1,n}$ and $M_{2,n}$ are non-incident non-maximal subgroups of G_n generating G_n. It is easy to verify that

$$|(M_{1,n}, M_{2,n})^i| = |(M_{2,n}, M_{1,n})^i| = 4^{n+1-i}$$

for $1 \leq i \leq n+1$. In particular, it follows that $(M_{1,n}, M_{2,n})^n \neq (M_{2,n}, M_{1,n})^n$.

Remark 1. Example 1 shows that the constant 6 in Theorems 1 and 2 cannot be decreased.

Remark 2. Example 2 shows that the constant 12 in the Corollary cannot be decreased.

Remark 3. In the Corollary, the condition of maximality of the subgroup M_1 in G is essential. As Example 3 shows, there exists a sequence of triples $(G_n, M_{1,n}, M_{2,n})$, $n \in \mathbb{N}$, such that G_n is a finite group, $M_{1,n}$ and $M_{2,n}$ are nonmaximal subgroups in G_n generating G_n, and $(M_{1,n}, M_{2,n})^n \neq (M_{2,n}, M_{1,n})^n$.

YET MORE STRONG VERSION OF THE SIMS CONJECTURE

Theorem 2 immediately follows from a stronger result, which we are planning to prove in a series of papers.

Let G, M_1, and M_2 satisfy the hypothesis of Theorem 2. We are interested in the case where $(M_1)_G = (M_2)_G = 1$ and $1 < |(M_1, M_2)^2| \leq |(M_2, M_1)^2|$. The set of all such triples (G, M_1, M_2) is denoted by Π. Consider triples from Π up to the following equivalence: the triples (G, M_1, M_2) and (G', M_1', M_2') from Π are equivalent if there exists an isomorphism of G on G' taking M_1 to M_1' and M_2 to M_2'.

The group G acts by conjugation faithfully and primitively on the set $X = \{gM_1g^{-1} \mid g \in G\}$. According to a refinement of the Thompson–Wielandt theorem (1970) for the case under consideration obtained by Fomin (Research in Group Theory, Sverdlovsk, 1990), the product

$$(M_1, M_2)^2(M_2, M_1)^2$$

is a nontrivial p-group for some prime p.

Depending on the form of the socle $Soc(G)$ of the group G, we partition the set Π into the following subsets:

Π_0 is the set of triples (G, M_1, M_2) from Π such that $Soc(G)$ is not a simple nonabelian group, i.e., G is not an almost simple group;

Π_1 is the set of triples (G, M_1, M_2) from Π with $Soc(G)$ isomorphic to an alternating group;

Π_2 is the set of triples (G, M_1, M_2) from $\Pi \setminus \Pi_1$ with $Soc(G)$ isomorphic to a simple group of Lie type over a field of a characteristic different from p;

Π_3 is the set of triples (G, M_1, M_2) from $\Pi \setminus (\Pi_1 \cup \Pi_2)$ with simple $Soc(G)$ isomorphic to a simple group of Lie type over a field of characteristic p;

Π_4 is the set of triples (G, M_1, M_2) from Π with $Soc(G)$ isomorphic to one of the 26 finite simple sporadic groups.

For a nonempty set Σ of triples (G, M_1, M_2), where G is a finite group and M_1 and M_2 are distinct conjugate maximal subgroups of G, define $c(\Sigma)$ to be the maximum positive integer c such that $(M_1, M_2)^{c-1} \neq 1$ or $(M_2, M_1)^{c-1} \neq 1$ for some triple $(G, M_1, M_2) \in \Sigma$. If such a maximum number does not exist, we set $c(\Sigma) = \infty$. Define $c(G, M_1, M_2) = c(\{(G, M_1, M_2)\})$ and $c(\varnothing) = 0$.

It was announced in (Dokl. Math. 59 (1999)) that $c(\Pi_0) \leq \max_{1 \leq i \leq 4} c(\Pi_i)$, $c(\Pi_1) = 0$, $c(\Pi_2) = 3$, $c(\Pi_3) = 6$, and $c(\Pi_4) = 5$. Theorem 2 follows from the equality $c(\Pi) = 6$.

Now we state the following problem which generalizes essentially Theorem 2 and can be considered as a yet more strong form of the Sims conjecture.

PROBLEM. Describe the set Π more precisely and find all triples from $\Pi \setminus \Pi_0$ up to equivalency.

The problem is of interest for finite group theory because the study of maximal subgroups is very important for finite group theory. Although local maximal subgroups of finite almost simple groups are now classified, their intersections are not sufficiently investigated. If G is a finite almost simple group and $(G, M_1, M_2) \in \Pi \setminus \Pi_0$, then M_1 and M_2 are some distinct conjugate local maximal subgroups in G whose intersection $M_1 \cap M_2$ is, in a sense, large.

The problem also is of interest for graph theory because a of it solution would give a description of all undirected connected finite graphs Γ whose a automorphism group G acts primitively on $V(\Gamma)$ and $G_x^{[2]} \neq 1$ for $x \in V(\Gamma)$.

For any $(G, M_1, M_2) \in \Pi$, let $\Gamma_{G,\{M_1,M_2\}}$ be the graph defined by

$$V(\Gamma_{G,\{M_1,M_2\}}) = \{gM_1g^{-1} : g \in G\},$$
$$E(\Gamma_{G,\{M_1,M_2\}}) = \{\{gM_1g^{-1}, gM_2g^{-1}\} : g \in G\}.$$

Let Γ be a connected finite graph and G a primitive group of automorphisms of Γ. Assume that $G_x^{[2]} \neq 1$ where $x \in V(\Gamma)$. Identify vertices of Γ with their stabilizers in G. Then $E(\Gamma)$ is the union of edge sets of some graphs of the form $\Gamma_{G,\{M_1,M_2\}}$, where $M_1 = G_x$, $M_2 = G_y$ for $\{x, y\} \in E(\Gamma)$ and $(G, M_1, M_2) \in \Pi$ (because $\Gamma_{G,\{M_1,M_2\}}$ is a subgraph of Γ and consequently the ball of radius 2 of Γ centered at x contains the ball of radius 2 of $\Gamma_{G,\{M_1,M_2\}}$ centered at x). In particular, assuming in addition that G is edge-transitive, we get that Γ coinsides with $\Gamma_{G,\{M_1,M_2\}}$, where $M_1 = G_x$, $M_2 = G_y$ for $\{x, y\} \in E(\Gamma)$ and $(G, M_1, M_2) \in \Pi$.

The aim of our series of papers is to solve the Problem.

In the first paper of this series (Trudy IMM UrO RAN 20, no. 4 (2014); translation in Proc. Steklov Inst. Math. 289, Suppl. 1 (2015)), we prove the following two theorems.

THEOREM 3 (REDUCTION THEOREM).
If $(G, M_1, M_2) \in \Pi_0$, then $Soc(G) = T^k$, where T is a simple nonabelian group, $k > 1$, and the inequality

$$c(G, M_1, M_2) \leq c(H, H_1, H_2)$$

holds for some group H such that $Soc(H) \cong T$ and some district conjugate maximal subgroups H_1 and H_2 of H. In particular, $c(\Pi_0) \leq \max_{1 \leq i \leq 4} c(\Pi_i)$.

THEOREM 4. The set Π_1 is empty and, consequently, $c(\Pi_1) = 0$.

In the second paper of the series (Trudy IMM UrO RAN 22, no. 2 (2016); translation in Proc. Steklov Inst. Math. 295, Suppl. 1 (2016))), we prove the following theorem.

THEOREM 5. Let $(G, M_1, M_2) \in \Pi_2$**,** $Soc(G)$ **be a simple group of exceptional Lie type and let** $M_1 \cap Soc(G)$ **be a non-parabolic subgroup of** $Soc(G)$**. Then** $(M_1, M_2)^3 = (M_2, M_1)^3 = 1$ **and one of the following holds:**

(a) $G \cong E_6^{\varepsilon}(r)$ **or** $G \cong E_6^{\varepsilon}(r) : 2$, $\varepsilon \in \{+, -\}$, $r \geq 5$ **is a prime,** $9|(r - \varepsilon 1)$, $(M_1, M_2)^2 = Z(O_3(M_1))$ **and** $(M_2, M_1)^2 = Z(O_3(M_2))$ **are elementary abelian groups of order** 3^3, $(M_1, M_2)^1 = O_3(M_1)$ **and** $(M_2, M_1)^1 = O_3(M_2)$ **are special groups of order** 3^6**, the group** $M_1/O_3(M_1)$ **is isomorphic to** $SL_3(3)$ **for** $G \cong E_6^{\varepsilon}(r)$ **and is isomorphic to** $GL_3(3)$ **for** $G \cong E_6^{\varepsilon}(r) : 2$**, the group** $M_1/O_3(M_1)$ **acts faithfully on** $O_3(M_1)/Z(O_3(M_1))$ **and induces the group** $SL_3(3)$ **on** $Z(O_3(M_1))$, $|Z(O_3(M_1)) \cap Z(O_3(M_2))| = 3^2$ **and** $M_1 \cap M_2 = N_{M_1 \cap Soc(G)}(Z(O_3(M_1)) \cap Z(O_3(M_2)))$;

(b) $G \cong Aut(^3D_4(2))$, $(M_1, M_2)^2 = Z(M_1)$ **and** $(M_2, M_1)^2 = Z(M_2)$ **are groups of order** 3**, non-contained in** $Soc(G)$, $M_1 \cong \mathbb{Z}_3 \times ((\mathbb{Z}_3 \times \mathbb{Z}_3) : SL_2(3))$, $(M_1, M_2)^1 = O_3(M_1)$, $(M_2, M_1)^1 = O_3(M_2)$ **and** $M_1 \cap M_2$ **is a Sylow** 3**-subgroup in** M_1**.**

In any case of items (a) **and** (b)**, the triples** (G, M_1, M_2) **from** Π **exist and form one class of equivalency.**

In the third paper of this series (Trudy IMM UrO RAN 22, no. 4 (2016); translation in Proc. Steklov Inst. Math. 299, Suppl. 1 (2017)), we prove the following theorem.

THEOREM 6. Let $(G, M_1, M_2) \in \Pi_2$, $Soc(G)$ **be a simple group of classical non-orthogonal Lie type and let** $M_1 \cap Soc(G)$ **be a non-parabolic subgroup in** $Soc(G)$. **Then**

$$(M_1, M_2)^3 = (M_2, M_1)^3 = 1$$

and one of the following holds:

(a) $G \cong Aut(L_3(3))$, $(M_1, M_2)^2 = Z(M_1)$ **and** $(M_2, M_1)^2 = Z((M_2)$ **are groups of order** 2, **non-contained in** $Soc(G)$, $M_1 \cong \mathbb{Z}_2 \times S_4$, $(M_1, M_2)^1 = O_2(M_1)$, $(M_2, M_1)^1 = O_2(M_2)$ **and** $M_1 \cap M_2$ **is a Sylow** 2**-subgroup in** M_1**;**

(b) $G \cong U_3(8) : 3_1$ **or** $U_3(8) : 6$, $(M_1, M_2)^2 = Z(M_1)$ **и** $(M_2, M_1)^2 = Z(M_2)$ **are groups of order** 3, **non-contained in** $Soc(G)$, $M_1 \cong \mathbb{Z}_3 \times (\mathbb{Z}_3^2 : SL_2(3))$ **or** $\mathbb{Z}_3 \times (\mathbb{Z}_3^2 : GL_2(3))$ $(M_1, M_2)^1 = O_3(M_1)$, $(M_2, M_1)^1 = O_3(M_2)$ **and** $M_1 \cap M_2$ **is a Sylow** 3**-subgroup in** M_1 **or its normalizer in** M_1, **respectively;**

(c) $G \cong L_4(3) : 2_2$ **or** $Aut(L_4(3))$, $(M_1, M_2)^2 = Z(M_1)$ **и** $(M_2, M_1)^2 = Z(M_2)$ **are groups of order** 2, **non-contained in** $Soc(G)$, $M_1 \cong \mathbb{Z}_2 \times S_4 \times S_4$ **or** $\mathbb{Z}_2 \times (S_4 \wr \mathbb{Z}_2)$, **respectively,** $(M_1, M_2)^1 = O_2(M_1)$, $(M_2, M_1)^1 = O_2(M_2)$ **and** $M_1 \cap M_2$ **is a Sylow** 2**-subgroup in** M_1.

In any case of items (a), (b) **and** (c), **the triples** (G, M_1, M_2) **from** Π **exist and form one class of equivalency.**

The description of Π_2 will be completed at our fourth paper, which is in preparation.

In particular, the following theorem is proved.

THEOREM 7. Let $(G, M_1, M_2) \in \Pi_2$**,** $Soc(G)$ **be a simple orthogonal group of the dimension** ≥ 7 **and** $M_1 \cap Soc(G)$ **be a non-parabolic subgroup in** $Soc(G)$**. Then** $Soc(G) \cong P\Omega_8^+(q)$**, where** q **is a prime power. Moreover if** q **is an odd prime, 16 divides** $q^2 - 1$**,** G **is a finite group with** $Soc(G) \cong P\Omega_8^+(q)$ **and** G **contains an element inducing on** $Soc(G)$ **a graph automorphism of order 3 (so-called triality) then there exists a triple** (G, M_1, M_2) **from** Π_2 **such that** $(M_1, M_2)^2 = Z(O_2(M_1))$ **and** $(M_2, M_1)^2 = Z(O_2(M_2))$ **are elementary abelian groups of order** 2^3**,** $(M_1, M_2)^1 = O_2(M_1)$ **and** $(M_2, M_1)^1 = O_2(M_2)$ **are special groups of order** 2^9**, the group** $M_1/O_2(M_1)$ **is isomorphic to** $PSL_3(2) \times \mathbb{Z}_3$ **or** $PSL_3(2) \times S_3$**, and** $M_1 \cap M_2$ **is a Sylow 2-subgroup in** M_1**.**

In subsequent papers of the series, the sets Π_3 and Π_4 will be described.

Triples from the set Π_3 form the majority of triples from the set $P \setminus \Pi_0$. We have many auxiliary results which will be useful for our investigation.

Let $(G, M_1, M_2) \in \Pi_3$. Then $L := Soc(G)$ is a simple group of Lie type over a finite field k of characteristic p and $M_0 := M_1 \cap L$ is a parabolic subgroup in L. We have $M_0 := U_0 L_0$ is a parabolic subgroup in G, where U_0 is the unipotent radical and L_0 is a Levi complement in M_0. We say that the group L is *special* if $p = 2$ for groups of type B_l, C_l, and F_4 and $p \leq 3$ for groups of type G_2. It follows from the results of H. Azad, M. Barry, and G. Seitz [Comm. Algebra, 18, no. 2 (1990)] that, for non-special groups L, factors of the lower central series of the group U are completely reducible kL_0-modules, decomposable as a direct product of chief factors of M_0. The number of these factors depends only on the Lie type of the group L, but not on k.

In 1997, I offered to my PhD student Vera V. Korableva the problem to study the structure and the pairwise intersections of conjugate parabolic maximal subgroups of finite simple groups of Lie type. This problem is closely connected with our problem for the set Π_3 and is of independent interest. Twenty years gone. She worked highly succesfully and defended dissertations of candidate (2000) and doctor (2011) of sciences.

In her papers [Trudy IMM UrO RAN 5 (1998); 7, no.2 (2001); 14, no. 4 (2008); 15, no. 2 (2009); 16, no. 3 (2010)], [Siberian Math. J. 49, no. 2 (2008), [In: Combinatorial and computational methods in mathematics, Omsk, 1999], [In; Low-dimensional topology and combinatorial group theory, Chelyabinsk, 1999], [VINITI, 1999], [Math. Notes 67, no. 1 (2000); 67, no. 6 (2000)], [Algebra and Logic 49, no. 3 (2010); 49, no. 5 (2010)], she obtained a description of intersections of two different conjugate parabolic maximal subgroups in all finite simple groups of Lie type.

Moreover, in recent papers [Siberian Math. J., 55, no. 4 (2014); 56, no. 5 (2015); 58 (2017)], [Trudy IMM UrO RAN 20, no. 2 (2014); translation in Proc. Steklov Inst. Math. 289, Suppl. 1 (2015], [Trudy IMM UrO RAN 21, no. 3 (2015)] Korableva obtained a refined description of chief factors of parabolic maximal subgroups involved in the unipotent radical for all groups of Lie type, including for special groups.