Generators for discrete subgroups of 2-by-2 matrices over rational quaternion algebras

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Goal: Generators for $\text{SL}_2(\mathbb{Z})$

- consider $M_2(\mathbb{Q})$ with $a, b < 0$
- order $(\frac{a,b}{\mathbb{Z}})$ in $(\frac{a,b}{\mathbb{Q}})$

Goal of this work
Finding generators for $(P)\text{SL}_2(\mathbb{Z})$.
Motivation: Units in Group Rings

- $G$ finite group.
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Open Problem: Finding a presentation for a subgroup of finite index of $\mathcal{U}(\mathbb{Z}G)$. 
Motivation: Units in Group Rings

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Open Problem: Finding a presentation for a subgroup of finite index of $U(\mathbb{Z}G)$.

- $\mathbb{Q}G = \prod_{i=1}^{n} M_{n_i}(D_i)$, $D_i$ a division algebra,
- let $O_i$ be an order in $D_i$ for every $1 \leq i \leq n$. 
Motivation: Units in Group Rings

- G finite group.

Open Problem: Finding a presentation for a subgroup of finite index of $\mathcal{U}(\mathbb{Z}G)$.

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- let $O_i$ be an order in $D_i$ for every $1 \leq i \leq n$.

Finding generators and relations for $\mathcal{U}(\mathbb{Z}G)$, up to commensurability, reduces to finding generators and relations for $\text{SL}_{n_i}(O_i)$ for every $1 \leq i \leq n$. 

Definition
A finite dimensional simple algebra is said to be an exceptional component, if it is one of the following types:

(1) a non-commutative division algebra different from a totally definite quaternion algebra,

(2) $M_2(\mathbb{Q})$,

(3) $M_2(\mathbb{Q}(\sqrt{-d}))$ with $d > 0$,

(4) $M_2(\mathcal{H})$ where $\mathcal{H}$ is a totally definite quaternion algebra with centre $\mathbb{Q}$, i.e. $\mathcal{H} = \left( \frac{a,b}{\mathbb{Q}} \right)$ with $a$ and $b$ negative integers.
Exceptional Components

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Idea for some of these cases $\rightarrow$ Discontinuous actions on hyperbolic spaces.
Isometries of $\mathbb{H}^2$ and $\mathbb{H}^3$

The upper half space model of hyperbolic space

- $\mathbb{H}^2 = \{ z = x + yi \mid x, y \in \mathbb{R}, y > 0 \}$
- $\mathbb{H}^3 = \{ z = x + yi + rj \mid x, y, r \in \mathbb{R}, r > 0 \}$

- $\text{PSL}_2(\mathbb{R}) \cong \text{ISO}^+(\mathbb{H}^2)$
- $\text{PSL}_2(\mathbb{C}) \cong \text{ISO}^+(\mathbb{H}^3)$

Action on $\mathbb{H}^2, \mathbb{H}^3$

- $\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$, computed in $\mathbb{C}$

- $\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = (az + b)(cz + d)^{-1}$, computed in $\left( \frac{-1}{\mathbb{R}}, -1 \right)$
Group Actions, Fundamental Domains and Poincaré

Theorem

Let $X$ be a proper metric space. A group $\Gamma$ of isometries of $X$ acts \textit{discontinuously} on $X$ if and only if it is a \textit{discrete} subgroup.
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**Definition**

A \textbf{fundamental domain} of the discontinuous group $\Gamma < \text{Iso}(X)$ is a closed subset $F \subseteq X$ satisfying the following conditions:

- the boundary of $F$ has Lebesgue measure 0,
- $g(F^\circ) \neq h(F^\circ)$ for $g \neq h$.
- $X = \bigcup_{g \in \Gamma} g(F)$.
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- $X = \bigcup_{g \in \Gamma} g(\mathcal{F})$.

Theorem (Poincaré)
Let $\mathcal{F}$ be a convex fundamental polyhedron for a discrete group $\Gamma$ of $\mathbb{H}^n$. Then $\Gamma$ is generated by

$$\{g \in \Gamma \mid \mathcal{F} \cap g(\mathcal{F}) \text{ is a side of } \mathcal{F}\}.$$
Fundamental Domain of $\text{PSL}_2(\mathbb{Z})$ acting on $\mathbb{H}^2$
Dirichlet Algorithm of Finite Covolume (DAFC)
joint work with E. Jespers, S. O. Juriaans, A. De A. E Silva, A. C. Souza Filho

\[ \Gamma \leq \text{PSL}_2(\mathbb{C}) \text{ discrete group of finite covolume} \]
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\[ \rightarrow \]

finite-sided convex polyhedron \( P \) of finite volume
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Γ ≤ PSL₂(ℂ) discrete group of finite covolume

▶ finite-sided convex polyhedron $P$ of finite volume
▶ $P$ contains fundamental domain for $Γ$
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\[ \Gamma \leq \text{PSL}_2(\mathbb{C}) \] discrete group of finite covolume

\[ \text{finite-sided convex polyhedron } P \] of finite volume

\[ P \text{ contains fundamental domain for } \Gamma \]

\[ \text{finite set of generators up to finite index for } \Gamma \]
Output for $\text{PSL}_2 \left( \mathbb{Z} \left[ \frac{1 + \sqrt{-23}}{2} \right] \right)$
What about $M_2(\frac{a,b}{Q})$?

- to begin: $M_2(\frac{-1,-1}{Q})$
- order: $\text{PSL}_2(\frac{-1,-1}{\mathbb{Z}})$

Main idea: imitate DAFC for $\Gamma \leq \text{PSL}_2(\frac{-1,-1}{\mathbb{R}})$ discrete
What about $M_2((\frac{a,b}{Q}))$?

- to begin: $M_2((\frac{-1,-1}{Q}))$
- order: $\text{PSL}_2((\frac{-1,-1}{\mathbb{Z}}))$

Main idea: imitate DAFC for $\Gamma \leq \text{PSL}_2((\frac{-1,-1}{\mathbb{R}}))$ discrete

What is $\text{PSL}_2((\frac{-1,-1}{\mathbb{R}}))$?

→ reduced norm 1
The action of $\text{PSL}_2\left(\left(\frac{-1,-1}{\mathbb{R}}\right)\right)$

- Möbius action on $\left(\frac{-1,-1}{\mathbb{R}}\right)$
- Action on $\mathbb{H}^5$ by Poincaré extension

\[\rightarrow \text{PSL}_2\left(\left(\frac{-1,-1}{\mathbb{R}}\right)\right) \cong \text{ISO}^+(\mathbb{H}^5).\]

However: not very handy to work with.
Clifford Algebras

Definition
The Clifford algebra $C_n$ is the associative algebra over the reals generated by elements $i_1, i_2, \ldots i_{n-1}$ satisfying

- $i_h^2 = -1$ for every $1 \leq h \leq n - 1$
- $i_hi_l = -i_l i_h$ for $h \neq l$.

Definition
The Clifford group $\Gamma_n$ is the group of all invertible elements of $C_n$. 
PSL_+(\Gamma_n) and its action on \( \mathbb{H}^{n+1} \)

\[
SL_+(\Gamma_n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad^* - bc^* = 1, + \text{ some conditions} \right\}
\]

Theorem (Ahlfors '81)
PSL_+(\Gamma_n) \cong ISO^+(\mathbb{H}^{n+1}).
in particular: PSL_+(\Gamma_4) \cong ISO^+(\mathbb{H}^5).

Möbius action
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = (az + b)(cz + d)^{-1}, \text{ computed in } C_{n+1}
\]

Main strategy: Imitate the algorithm for
PSL_+(\Gamma_4(\mathbb{Z})) \leq PSL_+(\Gamma_4), \text{ discrete.}
To sum up

\[
\text{PSL}_2\left(\left(-\frac{1}{\mathbb{R}}, -\frac{1}{\mathbb{R}}\right)\right) \cong \text{ISO}^+(\mathbb{H}^5) \cong \text{PSL}_+(\Gamma_4)
\]
To sum up

$$\text{PSL}_2((\mathbb{R}) \approx \text{ISO}^+(\mathbb{H}^5) \approx \text{PSL}_+(\Gamma_4)$$

$$\text{SL}_2(\Gamma_4(\mathbb{Q})) \sim \text{SL}_2((\mathbb{R}) \approx \text{SL}_2((\mathbb{Q})) \sim \text{SL}_2((\mathbb{Z}))$$
To sum up

$$\text{PSL}_2\left(\left(-1, -1\right) \mathbb{R}\right) \cong \text{ISO}^+(\mathbb{H}^5) \cong \text{PSL}_+\left(\Gamma_4\right)$$

\[ \forall I \quad \text{SL}_2\left(\Gamma_4(\mathbb{Q})\right) \sim \text{SL}_2\left(\left(-1, -1\right) \mathbb{Q}\right) \]

$$\forall I \quad \text{SL}_2\left(\Gamma_4(\mathbb{Z})\right) \text{ SL}_2\left(\left(-1, -1\right) \mathbb{Z}\right)$$

What about $\text{PSL}_2\left(\left(\frac{a,b}{\mathbb{Z}}\right)\right)$, $a, b < 0$?

$$\left(\frac{a,b}{\mathbb{R}}\right) \cong \left(\frac{-1,-1}{\mathbb{R}}\right)$$
Thank you for your attention.