Rearrangement Groups of Connected Spaces

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Rearrangement Groups

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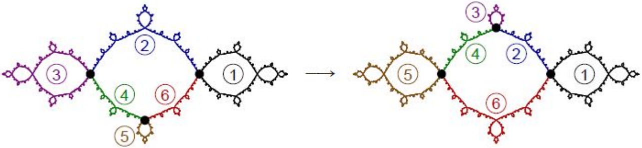

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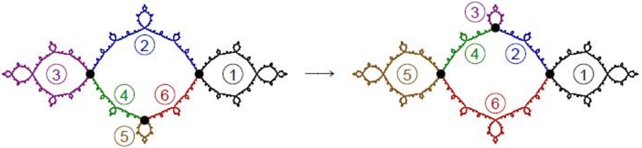
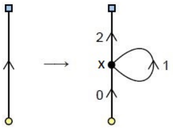
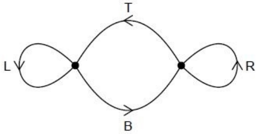


The topological space is constructed by performing edge replacements $e \to R$ on a finite directed graph G_0.





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An F-type rearrangement group has an infinite generating set that results from the geometric structure of the limit space.

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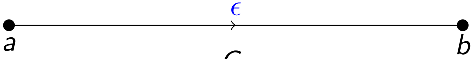

G_0

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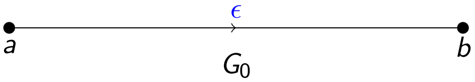

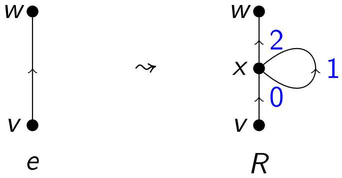

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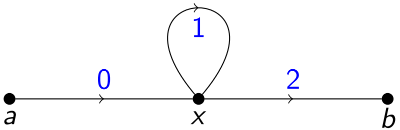

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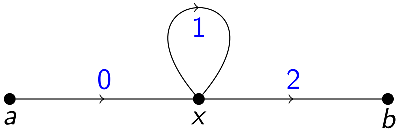

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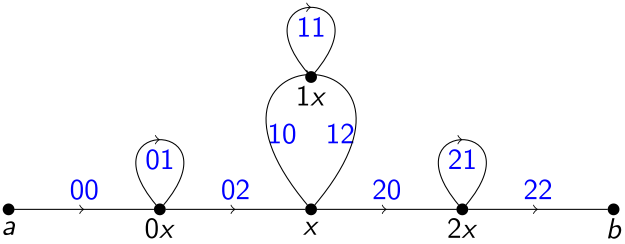
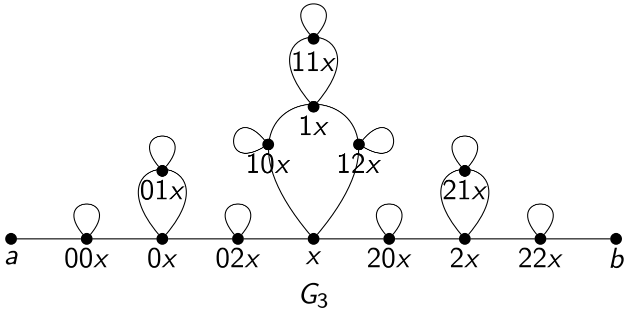


G_2

The Limit Space III



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Ω is the infinite product $E(G_0) \times E(R)^\infty$. Since it is the infinite product of finite sets, it is homeomorphic to the Cantor space.

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The *limit space* X is the quotient space $\Omega/\sim$.

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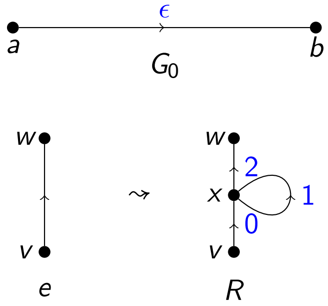

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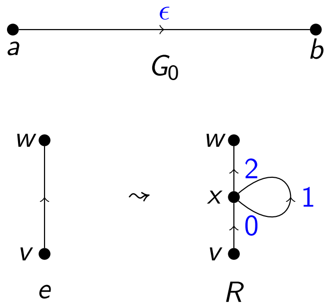

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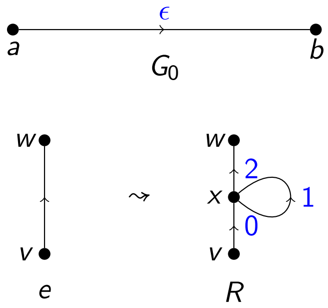

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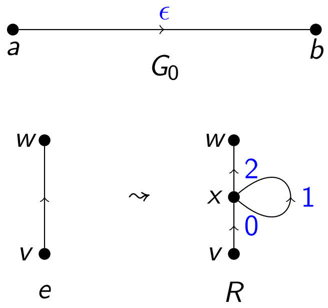

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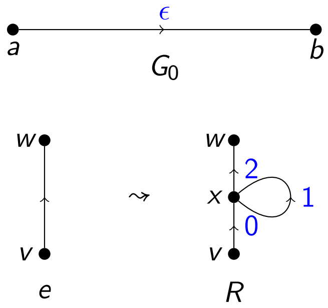

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- Neither G_0 nor R has any isolated vertices.
- The initial and terminal vertices of R are not connected by an edge.
- R has at least three vertices and two edges.

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More generally, if $e_0 \dots e_n x$ is any vertex of the graph G_{n+1}, then

$$e_0 \dots e_n 0\bar{2} \sim e_0 \dots e_n 1\bar{0} \sim e_0 \dots e_n 1\bar{2} \sim e_0 \dots e_n 2\bar{0}.$$

Cells

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Definition

Consider an edge $e_0 \dots e_n$ of G_n. We let $\Omega(e_0 \dots e_n)$ denote the set of all points in Ω that have $e_0 \dots e_n$ as a prefix. The cell $C(e_0 \dots e_n)$ is the image of $\Omega(e_0 \dots e_n)$ in the limit space X.

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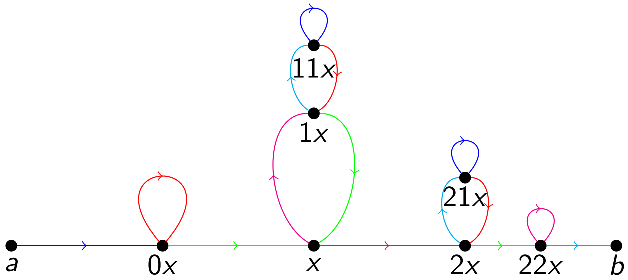
The cells of X can be represented by a rooted tree, since we can define an inclusion relation using prefixes.

That is,

- $C(e_0 \ldots e_n) \supseteq C(e'_0 \ldots e'_k)$ whenever $e_0 \ldots e_n$ is a prefix of $e'_0 \ldots e'_k$
- $C(e_0 \ldots e_n)$ and $C(e'_0 \ldots e'_k)$ have disjoint interiors if neither $e_0 \ldots e_n$ nor $e'_0 \ldots e'_k$ is a prefix of the other.

Cellular Partitions



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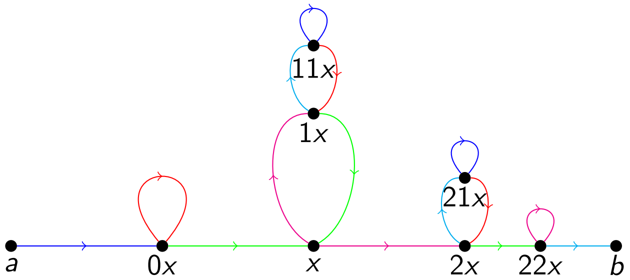

Definition

A cellular partition $\mathcal{P}$ of X is a cover of X by finitely many cells whose interiors are disjoint.

Cellular Partitions II

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- We can define a *lexicographic order* $<$.
- We can define a *depth order* $\prec$ by ordering them by depth.
- We can define a *level order* $\triangleleft$ by ordering them by how far removed they are from the line.

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Consider the cells $C(e_0 \dots e_n)$ and $C(e_0' \dots e_k')$ of X.

We can define a homeomorphism $\Phi\colon \Omega(e_0 \dots e_n) \to \Omega(e_0' \dots e_k')$ by

$$\Phi(e_0 \dots e_n \; e_{n+1} e_{n+2} \dots) = e_0' \dots e_k' \; e_{n+1} e_{n+2} \dots$$

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Definition

A homeomorphism $f\colon X \to X$ is called a *rearrangement* of X if there exists a cellular partition $\mathcal{P}$ of X such that f restricts to a canonical homeomorphism on each cell of P.

Rearrangements II

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Corresponding to a rearrangement f of X, there exist two cellular partitions $\mathcal{P}_{f,d}$ and $\mathcal{P}_{f,r}$ of X such that f maps each cell of $\mathcal{P}_{f,d}$ to a cell of $\mathcal{P}_{f,r}$. The set $(\mathcal{P}_{f,d}, \mathcal{P}_{f,r})$ is called a *cellular bipartition* for f.

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Theorem (Belk and Forrest, 2015)

The rearrangements of X form a group under composition.

The Generating Set for the F-Basilica Group

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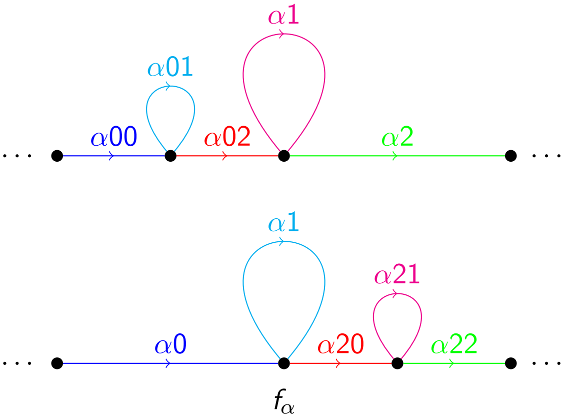
Proposition (K., Bleak, Quick)

The F-Basilica group is generated by the family f_α of rearrangements of X that act as follows on the cells with prefix α:



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Lemma

Consider a rearrangement f *of* X *and a cellular bipartition* $(\mathcal{P}_{f,d}, \mathcal{P}_{f,r})$ *for* f. *Then* f *does not change the level of a boundary point in* $\mathcal{P}_{f,d}$.

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Consider a rearrangement f of X and a cellular bipartition $(\mathcal{P}_{f,d}, \mathcal{P}_{f,r})$ for f. Then f preserves the lexicographic order of boundary points on the same level in $\mathcal{P}_{f,d}$.

Sketch of Proof II

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Lemma

Consider a rearrangement f of X and a cellular bipartition $(\mathcal{P}_{f,d}, \mathcal{P}_{f,r})$ for f. Consider a boundary point $z \in \mathcal{P}_{f,d}$ such that $\operatorname{depth} z < \operatorname{depth} zf$. Then there exists a rearrangement $f_\alpha^\pm$ such that

$$\operatorname{depth} zff_\alpha^\pm < \operatorname{depth} zf,$$

which does not impact the depth of the rest of the boundary points in $\mathcal{P}_{f,d}$.

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- What about rearrangement groups of disconnected spaces?
- What about other dynamical properties of rearrangement groups, such as conjugacy or centralizers?