On l^2-Betti numbers and their analogues in positive characteristic

Andrei Jaikin-Zapirain

Birmingham, August 12th, 2017

The initial setting

G is a finitely generated group.

$G > G_1 > G_2 > \ldots$ is a chain of normal subgroups of finite index with trivial intersection. In this setting G is residually finite.

K is a filed (of arbitrary characteristic), $A \in \mathrm{Mat}_{n\times m}(K[G])$

$$\begin{array}{cccc} \phi^A_{G/G_i}: & K[G/G_i]^n & \to & K[G/G_i]^m \\ & (v_1,\ldots,v_n) & \mapsto & (v_1,\ldots,v_n)A. \end{array}$$

$$\mathrm{rk}_{G/G_i}(A) = \frac{\dim_K \mathrm{Im}\,\phi^A_{G/G_i}}{|G:G_i|} = n - \frac{\dim_K \ker \phi^A_{G/G_i}}{|G:G_i|}$$

$\{\mathrm{rk}_{G/G_i}\}$ is a collection of Sylvester matrix rank functions on $K[G]$.

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Let R be a K-algebra. A Sylvester matrix rank function rk on R is a map $\mathrm{rk} : \mathrm{Mat}(R) \to \mathbb{R}_{\geq 0}$ satisfying the following conditions

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The main questions

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1. Is there the limit $\lim_{i\to\infty} \mathrm{rk}_{G/G_i}(A)$?
2. If the limit exists, how does it depend on the chain $\{G_i\}$?
3. What is the range of possible values for $\lim_{i\to\infty} \mathrm{rk}_{G/G_i}(A)$ for a given group G?

Conjectures

1. Yes, the limit exists.
2. It does not depend on the chain $\{G_i\}$.
3. Assume that there exists an upper bound for the orders of finite subgroups of G. Let

$$\mathrm{lcm}(G) = \mathrm{lcm}\{|H| : \ H \ \text{ is a finite subgroup of } G\}.$$

Then $\lim_{i\to\infty} \mathrm{rk}_{G/G_i}(A) \in \frac{1}{\mathrm{lcm}(G)}\mathbb{Z}$.

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Kaplansky's zero-divisor conjecture

Let G be a torsion-free group. Then the group ring K[G] does not contain nontrivial zero divisors, that is, it is a domain.

Conjectures 1 and 3 predict that if G is torsion-free ($\mathrm{lcm}(G) = 1$), then $\lim_{i\to\infty} \mathrm{rk}_{G/G_i}(A) \in \mathbb{Z}$.

$\mathrm{rk} = \lim_{i\to\infty} \mathrm{rk}_{G/G_i} \in \mathbb{P}(K[G])$ is faithful ($\mathrm{rk}(A) = 0$ iff $A = 0$)

P. Cohn: Assume that a K-algebra R has a faithful Sylvester matrix rank function taking only integer values. Then R can be embedded in a skew field.

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Let $\mathcal{C}$ be a CW-complex of dimension n and $\mathcal{C}_i$ $(0 \le i \le n)$ its i-dimensional cells.

Assume that G acts freely on $\mathcal{C}$ conserving the CW-structure and $G\backslash\mathcal{C}$ has only a finite number of cells.

Goal: we want to analyze $\lim\limits_{i\to\infty} \dfrac{dim_K H_p(G_i\backslash\mathcal{C}, K)}{|G:G_i|}$.

We use the cellular chain complex

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Motivation: the growth of the Betti numbers in a chain of coverings

Let $\mathcal{C}$ be a CW-complex of dimension n and $\mathcal{C}_i$ $(0 \leq i \leq n)$ its i-dimensional cells.
Assume that G acts freely on $\mathcal{C}$ conserving the CW-structure and $G\backslash\mathcal{C}$ has only a finite number of cells.

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Let Γ be an arithmetic subgroup of $SL_2(\mathbb{C})$ (e.g. $\Gamma = SL_2(\mathbb{Z}[i])$).
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F. Grunewald, A. Pinto, A. Jaikin, P. Zalesskii: $\mathrm{cd}(\mathcal{K}_\Gamma) \le 2$ and $\mathrm{cd}(\mathcal{K}_\Gamma) = 1$ if for every prime p and for every subgroup $G \le_f \Gamma$

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Let $G = G_1 > G_2 > \dots$ be a chain of normal subgroups with trivial intersection.

Conjectures (with coefficients in K)

- **L** (the Lück approximation conjecture over K)
  For every matrix A over $K[G]$, $\lim_{i\to\infty} \mathrm{rk}_{G/G_i}(A) = \mathrm{rk}_G(A)$.
- **A** (the strong Atiyah conjecture over K)
  Assume that there exists an upper bound for the orders of finite subgroups of G. For every matrix A over $K[G]$, $\mathrm{rk}_G(A) \in \frac{1}{\mathrm{lcm}(G)}\mathbb{Z}$.

$K \leq \mathbb{C}$:

Conjecture L $\Rightarrow$ Conjectures 1 and 2

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Conjecture L $\Rightarrow$ Conjectures 1 and 2

Conjectures L and A $\Rightarrow$ Conjecture 3

The class of elementary amenable groups is the smallest class of groups containing finite groups, abelian groups and closed under subgroups, extensions and direct unions.

	$K \leq \mathbb{C}$	$\mathrm{char} K > 0$
Conj. 1	Yes	Yes
Conj. 2	Yes	Yes
Conj. 3	Yes	Yes
Conj. L	Yes	X
Conj. A	Yes	X

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The class of elementary amenable groups is the smallest class of groups containing finite groups, abelian groups and closed under subgroups, extensions and direct unions.

	$K \leq \mathbb{C}$	$\mathrm{char} K > 0$
Conj. 1	Yes	Yes
Conj. 2	Yes	Yes
Conj. 3	Yes	Yes
Conj. L	Yes	X
Conj. A	Yes	X

The state of the conjectures

A finitely generated group G is amenable if there exists a family $\{F_i\}$ of finite subsets of G such that for any $g \in G$

$$\lim_{i\to\infty} \frac{|gF_i \cap F_i|}{|F_i|} = 1.$$

	$K \leq \mathbb{C}$	$\mathrm{char} K > 0$
Conj. 1	Yes Yes	Yes Yes
Conj. 2	Yes Yes	Yes Yes
Conj. 3	Yes ?	Yes ?
Conj. L	Yes Yes	X
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elamantary amenable amenable

The state of the conjectures

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	$K \leq \mathbb{C}$	$\mathrm{char} K > 0$
Conj. 1	Yes Yes	Yes Yes
Conj. 2	Yes Yes	Yes Yes
Conj. 3	Yes ?	Yes ?
Conj. L	Yes Yes	X
Conj. A	Yes ?	X

elemantary amenable amenable

residually torsion-free soluble groups; hyperbolic 3-orbifold groups; virtually special groups

	$K \leq \mathbb{C}$	$\mathrm{char} K > 0$
Conj. 1	Yes Yes Yes	Yes Yes ?
Conj. 2	Yes Yes Yes	Yes Yes ?
Conj. 3	Yes ? Yes	Yes ? ?
Conj. L	Yes Yes Yes	X
Conj. A	Yes ? Yes	X

elemantary amenable; amenable

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	$K \leq \mathbb{C}$	$\mathrm{char} K > 0$
Conj. 1	Yes Yes Yes	Yes Yes ?
Conj. 2	Yes Yes Yes	Yes Yes ?
Conj. 3	Yes ? Yes	Yes ? ?
Conj. L	Yes Yes Yes	X
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elemantary amenable; amenable

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	$K \leq \mathbb{C}$	$\mathrm{char} K > 0$
Conj. 1	Yes Yes Yes	Yes Yes ?
Conj. 2	Yes Yes Yes	Yes Yes ?
Conj. 3	Yes ? Yes	Yes ? ?
Conj. L	Yes Yes Yes	X
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elemantary amenable; amenable

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Conj. 1	Yes Yes Yes	Yes Yes ?
Conj. 2	Yes Yes Yes	Yes Yes ?
Conj. 3	Yes ? Yes	Yes ? ?
Conj. L	Yes Yes Yes	X
Conj. A	Yes ? Yes	X

elemantary amenable; amenable

A finitely generated group $G = \langle S \rangle$ is sofic if for any $\epsilon > 0$ and for any k there exists a finite S-labeled graph $X = (V, E)$ such that for at least $(1 - \epsilon)|V|$ vertices $v \in V$ of X, $B_k(v)$ is isomorphic (as a S-labeled graph) to $B_k(1_G)$ (a ball in the Cayley graph $Cay(G, S)$).

amenable and residually finite groups are sofic

	$K \leq \mathbb{C}$	$\mathrm{char} K > 0$
Conj. 1	Yes Yes Yes Yes	Yes Yes ? ?
Conj. 2	Yes Yes Yes Yes	Yes Yes ? ?
Conj. 3	Yes ? Yes ?	Yes ? ? ?
Conj. L	Yes Yes Yes Yes	X
Conj. A	Yes ? Yes ?	X

elemantary amenable; amenable; residually torsion-free soluble; hyperbolic 3-orbifold; virtually special

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amenable and residually finite groups are sofic

	$K \leq \mathbb{C}$	$\mathrm{char} K > 0$
Conj. 1	Yes Yes Yes Yes	Yes Yes ? ?
Conj. 2	Yes Yes Yes Yes	Yes Yes ? ?
Conj. 3	Yes ? Yes ?	Yes ? ? ?
Conj. L	Yes Yes Yes Yes	X
Conj. A	Yes ? Yes ?	X

elemantary amenable; amenable; residually torsion-free soluble; hyperbolic 3-orbifold; virtually special

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amenable and residually finite groups are sofic

	$K \leq \mathbb{C}$	$\mathrm{char} K > 0$
Conj. 1	Yes Yes Yes Yes	Yes Yes ? ?
Conj. 2	Yes Yes Yes Yes	Yes Yes ? ?
Conj. 3	Yes ? Yes ?	Yes ? ? ?
Conj. L	Yes Yes Yes Yes	X
Conj. A	Yes ? Yes ?	X

elemantary amenable; amenable; residually torsion-free soluble; hyperbolic 3-orbifold; virtually special

The state of the conjectures

	$K \leq \bar{\mathbb{Q}}$	$K \leq \mathbb{C}$	$\mathrm{char} K > 0$
Conj. 1	Yes[2] Yes[2] Yes[3] Yes[3]	Yes[2] Yes[2] Yes[6] Yes[6]	Yes[2] Yes[2] ? ?
Conj. 2	Yes[2] Yes[2] Yes[3] Yes[3]	Yes[2] Yes[2] Yes[6] Yes[6]	Yes[2] Yes[2] ? ?
Conj. 3	Yes[1] ? Yes[4] ?	Yes[1] ? Yes[5] ?	Yes[1] ? ? ?
Conj. L	Yes[2] Yes[2] Yes[3] Yes[3]	Yes[2] Yes[2] Yes[6] Yes[6]	X
Conj. A	Yes[1] ? Yes[4] ?	Yes[1] ? Yes[5] ?	X

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Conj. 1	Yes[2] Yes[2] Yes[3] Yes[3]	Yes[2] Yes[2] Yes[6] Yes[6]	Yes[2] Yes[2] ? ?
Conj. 2	Yes[2] Yes[2] Yes[3] Yes[3]	Yes[2] Yes[2] Yes[6] Yes[6]	Yes[2] Yes[2] ? ?
Conj. 3	Yes[1] ? Yes[4] ?	Yes[1] ? Yes[5] ?	Yes[1] ? ? ?
Conj. L	Yes[2] Yes[2] Yes[3] Yes[3]	Yes[2] Yes[2] Yes[6] Yes[6]	X
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The state of the conjectures

	$K \leq \bar{\mathbb{Q}}$	$K \leq \mathbb{C}$	$\mathrm{char} K > 0$
Conj. 1	Yes[2] Yes[2] Yes[3] Yes[3]	Yes[2] Yes[2] Yes[6] Yes[6]	Yes[2] Yes[2] ? ?
Conj. 2	Yes[2] Yes[2] Yes[3] Yes[3]	Yes[2] Yes[2] Yes[6] Yes[6]	Yes[2] Yes[2] ? ?
Conj. 3	Yes[1] ? Yes[4] ?	Yes[1] ? Yes[5] ?	Yes[1] ? ? ?
Conj. L	Yes[2] Yes[2] Yes[3] Yes[3]	Yes[2] Yes[2] Yes[6] Yes[6]	X
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	$K \leq \bar{\mathbb{Q}}$	$K \leq \mathbb{C}$	$\operatorname{char} K > 0$
Conj. 1	Yes[2] Yes[2] Yes[3] Yes[3]	Yes[2] Yes[2] Yes[6] Yes[6]	Yes[2] Yes[2] ? ?
Conj. 2	Yes[2] Yes[2] Yes[3] Yes[3]	Yes[2] Yes[2] Yes[6] Yes[6]	Yes[2] Yes[2] ? ?
Conj. 3	Yes[1] ? Yes[4] ?	Yes[1] ? Yes[5] ?	Yes[1] ? ? ?
Conj. L	Yes[2] Yes[2] Yes[3] Yes[3]	Yes[2] Yes[2] Yes[6] Yes[6]	X
Conj. A	Yes[1] ? Yes[4] ?	Yes[1] ? Yes[5] ?	X

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	$K \leq \bar{\mathbb{Q}}$	$K \leq \mathbb{C}$	$\mathrm{char} K > 0$
Conj. 1	Yes^2 Yes^2 Yes^3 Yes^3	Yes^2 Yes^2 Yes^6 Yes^6	Yes^2 Yes^2 ? ?
Conj. 2	Yes^2 Yes^2 Yes^3 Yes^3	Yes^2 Yes^2 Yes^6 Yes^6	Yes^2 Yes^2 ? ?
Conj. 3	Yes^1 ? Yes^4 ?	Yes^1 ? Yes^5 ?	Yes^1 ? ? ?
Conj. L	Yes^2 Yes^2 Yes^3 Yes^3	Yes^2 Yes^2 Yes^6 Yes^6	X
Conj. A	Yes^1 ? Yes^4 ?	Yes^1 ? Yes^5 ?	X

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	$K \leq \bar{\mathbb{Q}}$	$K \leq \mathbb{C}$	$\mathrm{char} K > 0$
Conj. 1	Yes[2] Yes[2] Yes[3] Yes[3]	Yes[2] Yes[2] Yes[6] Yes[6]	Yes[2] Yes[2] ? ?
Conj. 2	Yes[2] Yes[2] Yes[3] Yes[3]	Yes[2] Yes[2] Yes[6] Yes[6]	Yes[2] Yes[2] ? ?
Conj. 3	Yes[1] ? Yes[4] ?	Yes[1] ? Yes[5] ?	Yes[1] ? ? ?
Conj. L	Yes[2] Yes[2] Yes[3] Yes[3]	Yes[2] Yes[2] Yes[6] Yes[6]	X
Conj. A	Yes[1] ? Yes[4] ?	Yes[1] ? Yes[5] ?	X

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	$K \leq \bar{\mathbb{Q}}$	$K \leq \mathbb{C}$	$\mathrm{char} K > 0$
Conj. 1	Yes[2] Yes[2] Yes[3] Yes[3]	Yes[2] Yes[2] Yes[6] Yes[6]	Yes[2] Yes[2] ? ?
Conj. 2	Yes[2] Yes[2] Yes[3] Yes[3]	Yes[2] Yes[2] Yes[6] Yes[6]	Yes[2] Yes[2] ? ?
Conj. 3	Yes[1] ? Yes[4] ?	Yes[1] ? Yes[5] ?	Yes[1] ? ? ?
Conj. L	Yes[2] Yes[2] Yes[3] Yes[3]	Yes[2] Yes[2] Yes[6] Yes[6]	X
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	$K \leq \bar{\mathbb{Q}}$	$K \leq \mathbb{C}$	$\mathrm{char} K > 0$
Conj. 1	Yes[2] Yes[2] Yes[3] Yes[3]	Yes[2] Yes[2] Yes[6] Yes[6]	Yes[2] Yes[2] ? ?
Conj. 2	Yes[2] Yes[2] Yes[3] Yes[3]	Yes[2] Yes[2] Yes[6] Yes[6]	Yes[2] Yes[2] ? ?
Conj. 3	Yes[1] ? Yes[4] ?	Yes[1] ? Yes[5] ?	Yes[1] ? ? ?
Conj. L	Yes[2] Yes[2] Yes[3] Yes[3]	Yes[2] Yes[2] Yes[6] Yes[6]	X
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Conj. 1	Yes[2] Yes[2] Yes[3] Yes[3]	Yes[2] Yes[2] Yes[6] Yes[6]	Yes[2] Yes[2] ? ?
Conj. 2	Yes[2] Yes[2] Yes[3] Yes[3]	Yes[2] Yes[2] Yes[6] Yes[6]	Yes[2] Yes[2] ? ?
Conj. 3	Yes[1] ? Yes[4] ?	Yes[1] ? Yes[5] ?	Yes[1] ? ? ?
Conj. L	Yes[2] Yes[2] Yes[3] Yes[3]	Yes[2] Yes[2] Yes[6] Yes[6]	X
Conj. A	Yes[1] ? Yes[4] ?	Yes[1] ? Yes[5] ?	X

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The ideas of the proofs: an analytic approach.

Theorem (Lück (1993), [DLMSY, 2003], G. Elek, E. Szabo (2005))

Let G be a group and let $G = G_1 > G_2 > \dots$ be a chain of normal subgroups with trivial intersection. Assume G/G_i are sofic. Then for every matrix A over $\bar{\mathbb{Q}}[G]$, $\lim_{i\to\infty} \mathrm{rk}_{G/G_i}(A) = \mathrm{rk}_G(A)$.

1. We may assume that $A = BB^*$, whence ϕ^A_{G/G_i} and ϕ^A_G are selfadjoint positive operators
2. We associate measures μ^A_{G/G_i} with ϕ^A_{G/G_i} on an interval $[0, a]$. The theorem is equivalent to show that
$$\lim_{i\to\infty} \mu^A_{G/G_i}(0) = \mu^A_G(0)$$
3. μ^A_{G/G_i} weakly converges to μ^A_G, whence,
$$\limsup_{i\to\infty} \mu^A_{G/G_i}(0) \le \mu^A_G(0).$$
4. We use the conditions G/G_i are sofic and $K \le \bar{\mathbb{Q}}$ to show that $\mu^A_{G/G_i}(0, \epsilon)$ tends uniformly in i to 0 when ϵ tends to 0.

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The ideas of the proofs: an analytic approach.

Theorem (Lück (1993), [DLMSY, 2003], G. Elek, E. Szabo (2005))

Let G be a group and let $G = G_1 > G_2 > \dots$ be a chain of normal subgroups with trivial intersection. Assume G/G_i are sofic. Then for every matrix A over $\bar{\mathbb{Q}}[G]$, $\lim_{i\to\infty} \mathrm{rk}_{G/G_i}(A) = \mathrm{rk}_G(A)$.

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Main open problems

Problem 1

Extend the results from the characteristic 0 case to the characteristic $p > 0$ case.

Problem 2

Show that the strong Atiyah conjecture holds for one-relator groups.

one-relator groups with torsion are virtually special

Problem 3

Show that the strong Atiyah conjecture holds for subgroups of $\mathrm{GL}_n(\mathbb{C})$.

If G is a f.g. subgroup of $\mathrm{GL}_n(\mathbb{C})$ then it is known that there exists $H <_f G$ such that H satisfies the strong Atiyah conjecture.

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THANK YOU FOR YOUR ATTENTION