The Strong Symmetric Genus of Almost All $D$-type Generalized Symmetric Groups

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Strong Symmetric Genus

**Definition**

Given a finite group $G$, the smallest genus of any closed orientable topological surface on which $G$ acts faithfully as a group of orientation preserving symmetries is called the **strong symmetric genus** of $G$. 

If $\sigma_0(G) > 1$ for a finite group $G$, then $\sigma_0(G) \geq 1 + |G|/84$. We have equality if $G$ is a Hurwitz group.
Strong Symmetric Genus

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- The strong symmetric genus of the group $G$ is denoted $\sigma^0(G)$.
- If $\sigma^0(G) > 1$ for a finite group $G$, then $\sigma^0(G) \geq 1 + \frac{|G|}{84}$.
- We have equality if $G$ is a Hurwitz group.
Known results on the strong symmetric genus

- All groups $G$ such that $\sigma^0(G) \leq 25$ are known. [Broughton, 1991; May and Zimmerman, 2000 and 2005; Fieldsteel, Lindberg, London, Tran and Xu, (Advised by Breuer) 2008]

- For each positive integer $n$, there is exists a finite group $G$ with $\sigma^0(G) = n$. [May and Zimmerman, 2003]
The strong symmetric genus is known for the following groups:

- $PSL_2(q)$ [Glover and Sjerve, 1985 and 1987]
- $SL_2(q)$ [Voon, 1993]
- the sporadic finite simple groups [Conder, Wilson and Woldar, 1992; Wilson, 1993, 1997 and 2001]
- alternating and symmetric groups [Conder, 1980 and 1981]
- the hyperoctahedral groups [J, 2004]
- the remaining finite Coxeter groups [J, 2007]
- the generalized symmetric groups of type $G(n,3)$ [J, 2010]
Generators and the Riemann-Hurwitz Equation

- If a finite group $G$ has generators $x$ and $y$ of orders $p$ and $q$ respectively with $xy$ having the order $r$, then we say that $(x, y)$ is a $(p, q, r)$ generating pair of $G$. 

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Strong Symmetric Genus of $D$-type Groups
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- For ease of comparison we will assume that $p \leq q \leq r$. Note that a $(p, q, r)$ generating pair also yields a $(q, p, r)$ generating pair and the like.
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- For ease of comparison we will assume that $p \leq q \leq r$. Note that a $(p, q, r)$ generating pair also yields a $(q, p, r)$ generating pair and the like.
- The existence of a $(p, q, r)$ generating pair gives a faithful orientation preserving action of the group $G$ on a surface $S$. 
Generators and the Riemann-Hurwitz Equation

- The existence of a \((p, q, r)\) generating pair gives a faithful orientation preserving action of the group \(G\) on a surface \(S\).
- This is done by realizing the group \(G\) as a quotient of the triangle group

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\Delta(p, q, r) = \langle x, y | x^p = y^q = (xy)^r = 1 \rangle.
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Generators and the Riemann-Hurwitz Equation

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- The genus of the surface \(S\) is then found from the Riemann-Hurwitz formula:

\[
\text{genus}(S) = 1 + \frac{|G|}{2} \left(1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}\right).
\]
A \((p, q, r)\) generating pair of \(G\) is called a minimal generating pair if no generating pair for the group \(G\) gives an action on a surface of smaller genus.

For the groups we will be working with \(\sigma^0(G) \geq 2\) or equivalently any generating pair will be a \((p, q, r)\) generating pair with \(\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1\).
Minimal Generating Pairs

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The Riemann-Hurwitz formula:

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A Lemma by Singerman

Lemma (Singerman)

Let $G$ be a finite group such that $\sigma^0(G) > 1$. If $|G| > 12(\sigma^0(G) - 1)$, then $G$ has a $(p, q, r)$ generating pair with

$$\sigma^0(G) = 1 + \frac{1}{2}|G| \cdot \left(1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}\right).$$
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- Singerman’s Lemma implies that if $G$ has a minimal $(p, q, r)$ generating pair such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \geq \frac{5}{6}$, then the strong symmetric genus is given by this generating pair.
- Since $\sigma^0(G) > 1$, we know that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. 

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Strong Symmetric Genus of $D$-type Groups
More on Singerman’s Lemma

- Recall: if $G$ has a minimal $(p, q, r)$ generating pair such that $\frac{5}{6} \leq \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, then the strong symmetric genus is given by this generating pair.

- The triples of numbers $(p, q, r)$ that fit this requirement are:
  - $(2, 3, r)$ for any $r \geq 7$.
  - $(2, 4, r)$ for $5 \leq r \leq 11$.
  - $(3, 3, r)$ for $r = 4$ or $r = 5$.
More on Singerman’s Lemma

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- The groups in this talk have $S_n$ as a subgroup. So at least two numbers in the triple must be of even.

- The triples fitting both requirements are:
  - $(2, 3, r)$ for $r \geq 8$ even.
  - $(2, 4, r)$ for $5 \leq r \leq 11$.  

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Strong Symmetric Genus of $D$-type Groups
Generalized Symmetric Groups

- $G(n, m) = \mathbb{Z}_m \wr S_n$ for $n > 1$ and $m \geq 1$. 
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  - the permutation matrices and
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  - the diagonal matrices with entries in a multiplicative cyclic group of size $m$.
- $G(n, 1)$ is the symmetric group $S_n$.
- $G(n, 2)$ is the hyperoctahedral group $B_n$.
- The strong symmetric genus has been found for the groups:
  - $G(n, 1)$ [Conder, 1980]
  - $G(n, 2)$ and $G(n, 3)$ [J, 2004 and 2010]
  - $G(3, m)$, $G(4, m)$ and $G(5, m)$ [Ginter, Johnson, McNamara, 2008]
$D(n, m) = (\mathbb{Z}_m)^{n-1} \rtimes S_n$ for $n > 2$ and $m \geq 1$. 
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**D-type Generalized Symmetric Groups**

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- $D(n, m)$ is an index $m$ subgroup of $G(n, m)$.
- $D(n, m)$ is the smallest group of $n \times n$ matrices containing
  - the permutation matrices and
  - the diagonal matrices with entries in a multiplicative cyclic group of size $m$ each having determinant 1.
\[ D(n, m) = (\mathbb{Z}_m)^{n-1} \rtimes S_n \text{ for } n > 2 \text{ and } m \geq 1. \]

- \( D(n, m) \) is an index \( m \) subgroup of \( G(n, m) \).
- \( D(n, m) \) is the smallest group of \( n \times n \) matrices containing
  - the permutation matrices and
  - the diagonal matrices with entries in a multiplicative cyclic group of size \( m \) each having determinant 1.

- The strong symmetric genus has been found for the groups \( D(n, 2) \) which are the finite Coxeter groups of type \( D \) [J, 2007]
- We will be looking at the groups \( D(n, m) \) for \( m > 2 \).
Notation for elements of $D(n, m)$

- Recall that the group $D(n, m) = (\mathbb{Z}_m)^{n-1} \rtimes S_n$.
- An element of $D(n, m)$ will be denoted by $[\sigma, a]$ where
  - $\sigma$ is an element of $S_n$, and
  - $a$ is an element of $(\mathbb{Z}_m)^{n-1}$, which we will think of as a list of $n$ integers modulo $m$ such that the sum of the list is congruent to 0 modulo $m$.
- Notice that multiplication in the group is given by
  \[
  [\sigma, a] \cdot [\tau, b] = [\sigma \cdot \tau, \tau^{-1}(a) + b]
  \]
  where $\tau^{-1}$ is acting on the list $a$ and the addition is term by term modulo $m$. 
Suppose that $S_n$ is generated by two elements $\sigma$ and $\tau$ such that

- The number $m > 2$ divides the order of $\sigma$, and
- $\sigma$ has two fixed points.
- If $m$ and $n$ are even then $\sigma$ must have a third fixed point.
New generators from old

Suppose that $S_n$ is generated by two elements $\sigma$ and $\tau$ such that

- The number $m > 2$ divides the order of $\sigma$, and
- $\sigma$ has two fixed points.
- If $m$ and $n$ are even then $\sigma$ must have a third fixed point.

Then $[\sigma, a]$ and $[\tau, b]$ generate $D(n, m)$ where

- $b$ is a list of zeros,
- $a$ is a list where one fixed point of $\sigma$ has a 1 and the other fixed point has a -1,
- the rest of $a$ is filled in so that the elements permuted by each cycle of $\sigma$ add to zero modulo $m$ and the elements permuted by each cycle of $\tau \cdot \sigma$ add to zero modulo $m$. 
Suppose that $S_n$ is generated by two elements $\sigma$ and $\tau$ such that

- $3|m$, $9 \nmid m$, and the number $s = \frac{m}{3}$ divides the order of $\sigma$,
- $\tau$ has order 3, and
- both $\sigma$ and $\tau$ have two fixed points.
- If $m$ and $n$ are even then $\sigma$ must have a third fixed point.
Then $[\sigma, a]$ and $[\tau, b]$ generate $D(n, m)$ where

- $a$ is a list where one fixed point of $\sigma$ has a 3 and the other fixed point has a -3,

- $b$ is a list where one fixed point of $\tau$ has an $s$ and the other has a number $-s$, and

- the rest of $a$ and $b$ are filled in so that each of the following add to 0 modulo $m$:  
  - the elements of $a$ permuted by each cycle of $\sigma$
  - the elements of $b$ permuted by each cycle of $\tau$, and
  - the elements of $\sigma^{-1}(b) + a$ permuted by each cycle of $\tau \cdot \sigma$ add to zero modulo $m$.  

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Strong Symmetric Genus of $D$-type Groups
Orders

- Given the $\sigma$ and $\tau$ that generate $S_n$ and satisfy the conditions from either of the past two slides
- the new elements that we created $[\sigma, a]$ and $[\tau, b]$ generate $D(n, m)$.
- In addition the orders of $[\sigma, a],[\tau, b]$ and

  $$[\tau, b] \cdot [\tau, b] = [\tau \cdot \sigma, \sigma^{-1}(b) + a]$$

  are the same as $\sigma$, $\tau$ and $\tau \cdot \sigma$, respectively.
Given an integer $m > 2$ define $r(m)$ using the following criteria:

- If $m = 3, 4, \text{ or } 6$, then $r(m) = 8$
- If $m = 12$, then $r(m) = 12$.
- If $3 | m$ but $9 \not| m$ then
  - let $r(m) = \frac{m}{3}$ for $m$ even and $r(m) = \frac{2m}{3}$ for $m$ odd.
- Otherwise let $r(m) = m$ for $m$ even and $r(m) = 2m$ for $m$ odd.
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- If $m = 3, 4, \text{ or } 6$, then $r(m) = 8$
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- If $3|m$ but $9 \nmid m$ then let $r(m) = \frac{m}{3}$ for $m$ even and $r(m) = \frac{2m}{3}$ for $m$ odd.
- Otherwise let $r(m) = m$ for $m$ even and $r(m) = 2m$ for $m$ odd.

Notice that

- for all $m$, $m|3r(m)$,
- if $3 \nmid m$ or $9|m$, then $m|r(m)$, and
- $r(m)$ is always even.
Conder’s Generators

We use Conder’s Papers “More on generators for alternating and symmetric groups” Quart. J. Math. Oxford (2), 32 (1981) 137-163. Using the coset diagrams from the paper, we see that given $m > 2$ there are generators $\sigma$ and $\tau$ for all but finitely many symmetric groups $S_n$ such that

- $\sigma$ has order $r(m)$,
- $\tau$ has order 3,
- $\sigma$ has three fixed points, and $\tau$ has two fixed points.
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- $\sigma$ has order $r(m),$
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- $\sigma$ has three fixed points, and $\tau$ has two fixed points.

For a fixed $m$, this allows for the creation of a $(2, 3, r(m))$ generating pair for all but finitely many $D(n, m)$.

We are left to show that these generators are a minimal generating pair.
Other Generators

To claim that our generators are a minimal generating pair, we need to show that there cannot be a generating pair with a better \((p, q, r)\) triple.
Other Generators

- To claim that our generators are a minimal generating pair, we need to show that there cannot be a generating pair with a better \((p, q, r)\) triple.
- If any prime power \(p^i\) which divides \(m\) does not divide \(q\) or \(r\), then \(D(n, m)\) cannot have a \((2, q, r)\) generating pair.
To claim that our generators are a minimal generating pair, we need to show that there cannot be a generating pair with a better \((p, q, r)\) triple.

If any prime power \(p^i\) which divides \(m\) does not divide \(q\) or \(r\), then \(D(n, m)\) cannot have a \((2, q, r)\) generating pair.

The best (hyperbolic) triple not of the form \((2, q, r)\) where two of the three numbers are even is \((3, 4, 4)\).

Notice that

\[
\frac{1}{2} + \frac{1}{3} + \frac{1}{r(m)} > \frac{5}{6} = \frac{1}{3} + \frac{1}{4} + \frac{1}{4}.
\]
Exceptions

- The triples left that could be better are \((2, q, r)\) with \(m|qr\) and \((q, r) = 1\).
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- If \(q \leq r\) and \(\frac{1}{2} + \frac{1}{q} + \frac{1}{r} < 1\), the triples to consider are \((2, 4, r)\) for \(r \geq 5\).
Exceptions

- The triples left that could be better are \((2, q, r)\) with \(m \mid qr\) and \((q, r) = 1\).
- If \(q \leq r\) and \(\frac{1}{2} + \frac{1}{q} + \frac{1}{r} < 1\), the triples to consider are \((2, 4, r)\) for \(r \geq 5\).
- Checking sums of reciprocals leaves two cases,
  - \(m = 20\) and the triple \((2, 4, 5)\), and
  - \(m = 28\) and the triple \((2, 4, 7)\).
Exceptions

- The triples left that could be better are \((2, q, r)\) with \(m|qr\) and \((q, r) = 1\).
- If \(q \leq r\) and \(\frac{1}{2} + \frac{1}{q} + \frac{1}{r} < 1\), the triples to consider are \((2, 4, r)\) for \(r \geq 5\).
- Checking sums of reciprocals leaves two cases,
  - \(m = 20\) and the triple \((2, 4, 5)\), and
  - \(m = 28\) and the triple \((2, 4, 7)\).
- It turns out that in these two cases the \((2, 4, r)\) triple has a generating pair for all but finitely many cases.
Exceptions - Solved

- We used Brett Everitt’s paper "Permutation Representations of the (2, 4, r) triangle groups."
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Exceptions - Solved

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- This paper does not consider the case (2, 4, 5) since that work had been done earlier by Graham Higman.
- With a slight modification to the coset diagrams in this paper and a similar process to what we did in the (2, 3, r(m)) case, we create a (2, 4, 7)-generating pair for all but finitely many of the $D(n, 28)$ groups.
Exceptions - Solved

- This leaves just the case where $m = 20$.
- The coset diagrams for the $(2, 4, 5)$-generating pairs for all but finitely many of the groups $S_n$ was unpublished work.
Exceptions - Solved

- This leaves just the case where $m = 20$.
- The coset diagrams for the $(2, 4, 5)$-generating pairs for all but finitely many of the groups $S_n$ was unpublished work.
- Therefore we created our own collection of coset diagrams which give appropriate generators for all but finitely many $S_n$.
- As in earlier cases this $(2, 4, 5)$-generating pair of $S_n$ can be modified to be a $(2, 4, 5)$-generating pair of $D(n, 20)$.
Theorem

Given a fixed $m > 2$, where $m$ is neither 20 or 28, for all but finitely many positive integers $n$, the $D$-type generalized symmetric group $D(n, m)$ has a $(2, 3, r(m))$-minimal generating pair. In addition all but finitely many of the groups $D(n, 20)$ have a $(2, 4, 5)$-minimal generating pair and all but finitely many of the groups $D(n, 28)$ have a $(2, 4, 7)$-minimal generating pair.
Theorem

Given a fixed $m > 2$, where $m$ is neither 20 or 28, for all but finitely many positive integers $n$

$$\sigma^0(D(n, m)) = \frac{n!m^{n-1}(r(m) - 6)}{12r(m)} + 1.$$  

In addition for all but finitely many positive integers $n$

$$\sigma^0(D(n, 20)) = \frac{n!m^{n-1}}{40} + 1 \text{ and } \sigma^0(D(n, 28)) = \frac{3n!m^{n-1}}{56} + 1.$$