Depth of subgroups in finite groups

Erzsébet Horváth

Department of Algebra, Budapest University of Technology and Economics, Hungary

Groups St Andrews 2017 in Birmingham: 10th August 2017

Main question of this talk

We will consider a question by Lars Kadison:

Are there subgroups of even depth > 6?

History

The notion of depth originates from von-Neumann algebras and Hopf algebras. Later introduced for group algebras.

Several depth concepts: combinatorial, ordinary, modular

Recent papers on it: by S. Burciu, L. Kadison, B. Külshammer, R. Boltje, S. Danz, T. Fritzsche and C. Reiche.

Our work with L. Héthelyi and F. Petényi:

In the Suzuki groups $Sz(q)$ and Ree groups $R(q)$ determined the combinatorial and ordinary depth of maximal subgroups.
(These papers can be found in my homepage.)

Ordinary depth, inclusion matrix

Possible ways to define ordinary depth: with tensor products, with the inclusion matrix, with distance of characters

Inclusion matrix:
G, finite group $H \leq G$

$$\mathrm{Irr}(G) = \{\chi_1, \dots, \chi_k\}$$

$$\mathrm{Irr}(H) = \{\phi_1, \dots, \phi_r\}$$

$M := (m_{i,j}) \in \mathbb{Z}^{r \times k}$, where $m_{i,j} = (\phi_i^G, \chi_j) = (\phi_i, \chi_{j\,H})$.

then M is the inclusion matrix or Frobenius matrix of $H \leq G$.

„Powers" of M, the entries of „powers" of M

Define the „Powers" of M:

$M^{(1)} := M$, $M^{(2)} = MM^T$, $M^{(2i)} := (MM^T)^i \in \mathbb{Z}^{r\times r}$,
$M^{(2i+1)} := M^{(2i)}M \in \mathbb{Z}^{r\times k}$.

The entries of „powers" of M are:

$M^{(2)}_{i,j} = (\phi_i^G, \phi_j^G) = (Res_H^G Ind_H^G \phi_i, \phi_j)$,
$M^{(2m)}_{i,j} = ((Res_H^G Ind_H^G)^m \phi_i, \phi_j)$ and
$M^{(2m+1)}_{i,j} = (Ind_H^G (Res_H^G Ind_H^G)^m \phi_i, \chi_j)$.

We note:

$M^{(2)}_{i,j} \neq 0$ iff $\exists \chi \in \mathrm{Irr}(G)$ s.t. $(\chi_H, \phi_i) \neq 0$ and $(\chi_H, \phi_j) \neq 0$.

Distance of characters

$\phi, \psi \in \mathrm{Irr}(H)$ are related $\phi \sim \psi$ if $\exists \chi \in \mathrm{Irr}(G)$ s.t. $(\chi_H, \phi) \neq 0$ and $(\chi_H, \psi) \neq 0$.
We define the distance of irreducible characters of H:

1. $d(\phi, \phi) := 0$,
2. $d(\phi, \psi) := 1$ if $\phi \neq \psi$ and $\phi \sim \psi$.
3. $d(\phi, \psi) := m$ if there is a chain of irreducible characters: $\phi = \phi_0 \sim \phi_1 \sim \cdots \sim \phi_m = \psi$ and no shorter chain exists.
4. $d(\phi, \psi) := -\infty$ if there is no chain between ϕ and ψ.

We note: $M_{i,j}^{(2)} \neq 0$ iff $d(\phi_i, \phi_j) \leq 1$.

$M_{i,j}^{(2m)} \neq 0$ and $M_{i,j}^{(2m-2)} = 0$ iff $d(\phi_i, \phi_j) = m$.

Depth of the inclusion matrix

Let $H \leq G$ and let $M = (m_{i,j})$ be its inclusion matrix.

The depth of M is

$$d(M) := \min\{i \geq 1 | \exists q > 0, M^{(i+1)} \leq qM^{(i-1)}\}.$$

$$= \min\{i \geq 1 | Z(M^{(i-1)}) = Z(M^{(i+1)})\},$$

where Z denotes the set of zero positions.

Depth of group inclusion

Let $H \leq G$ then the ordinary depth of group inclusion is $d(H, G) := d(M)$, where M is the inclusion matrix of $H \leq G$. We note that:

1. $d(H, G) = 1$ iff $G = HC_G(x)$ for every $x \in H$.
2. $d(H, G) \leq 2$ iff $H \lhd G$.

Open problem: Characterize in group theoretical way that $d(H, G) = m$ for $m > 2$.

Remark: If $\exists x$ such that $H^x \cap H = 1$ then $d(H, G) = 3$.

The converse is not true: e.g. see $G = D_{12}$ and $H = C_2 \times C_2$.

Some results on depth, examples

Burciu, Kadison and Külshammer proved:

1. $d(H,G) \leq 2m+1$ iff $d(\phi_i,\phi_j) \leq m$ for every $\phi_i,\phi_j \in \mathrm{Irr}(H)$.
2. $d(H,G) \leq 2m$ iff $\max_{\chi\in\mathrm{Irr}(G)} \max_{\alpha\in\mathrm{Irr}(H)} d(\alpha,\chi_H) \leq m-1$.
3. If $N = Core_G(H) = \cap_{i=1}^{m} H^{x_i}$ then $d(H,G) \leq 2m$, if $N \leq Z(G)$ also holds then $d(H,G) \leq 2m-1$.

The smallest examples of subgroups of even depth > 2 are:

1. $d(D_8, S_4) = 4$
2. $d(C_4 \times C_4, C_2 \times ((C_4 \times C_4) : C_3))) = 6$

(In the second example a non-normal $C_4 \times C_4$ subgroup is taken).

Kadison's and Héthelyi's question

Lars Kadison asked (on his homepage) if there exist group inclusions of even depth > 6 ?
Our answer is yes. We have found depth 8 group inclusions.

Still open: Are there examples of depth 10 or bigger even depth?
Can even depth be arbitrarily large?

Odd depth can be arbitrarily large since $d(S_n, S_{n+1}) = 2n - 1$.

Laci Héthelyi asked:
Are the depths of maximal subgroups of simple groups always odd?

We observed that in $Sz(q)$ and $R(q)$ this is true:
3 and 5 were the values of depth of maximal subgroups.

Answer to Héthelyi's question

There exist simple groups with maximal subgroups of even depth.
Let us consider Alternating groups of degree $n \geq 5$:

1. A_5- depths of proper nontrivial subgroups are: 3, 5.
2. A_6- depths of proper nontrivial subgroups are: 3, 4, 5.
   depth 4: two conjugacy classes of maximal subgroups, both isomorphic to S_4.
3. A_7- depths of nontrivial proper subgroups are: 3, 5, 7.
4. A_8- depths of nontrivial proper subgroups are: 3, 5, 6, 7, 9.
   depth 6: unique up to conjugacy and maximal,
   $\simeq 2^4 : (GL(2,2) \times GL(2,2))$. (Parabolic in $GL(4,2) \simeq A_8$).
5. A_9- depths of nontrivial proper subgroups are: 3, 5, 6, 11.
   depth 6: unique up to conjugacy and maximal, $\simeq S_7$.

Depth of subgroups in A_n, $n \geq 10$

1. $A_{10}-$ depths of maximal subgroups are odd.
   The depths of nontrivial proper subgroups are: 3, 4, 5, 7, 13.
   depth 4: unique up to conjugacy, $\simeq C_2 \times S_6$.
2. $A_{11}-$ depths of proper subgroups are: 3, 5, 7, 13.
3. $A_{12}-$ the depths of maximal subgroups are: 4, 5, 7, 9, 15.
   depth 4: unique up to conjugacy, $\simeq S_3 \wr S_4$.
4. $A_{13}-$ depths of maximal subgroups are: 3, 7, 9, 17.
5. $A_{14}-$ depths of maximal subgroups are: 3, 7, 11, 19.
6. $A_{15}-$ depths of maximal subgroups are: 3, 5, 7, 8, 11, 21.
   depth 8: unique up to conjugacy, $\simeq A_{15} \cap (S_{12} \times S_3)$.
7. $A_n, n \in \{16, \ldots, 23\}-$ no even depth maximal subgroups.

Note: above subgroups have the same depths in S_n as in A_n.

Other depth 8 subgroups

Looking at groups of the ATLAS one finds that $O_8^-(2)$ also has depth 8 subgroups of structure $2^6 : U_4(2)$.

Looking at the iterated wreath product $G := ((C_2 \wr C_2) \wr C_2) \wr C_2$ we can also find some subgroups of depth 8, e.g. consider $G \cap (A_8 \times A_8)$ inside this group. Among its 63 maximal subgroups up to conjugacy there are 24 of depth 8 in G.

For the construction of these examples, the GAP system was used.

Thank you for your attention.

Acknowledgement

Supported by NKFI-Grant No.115288 and 115799.