Schur Multiplier of Central Product of Groups

Sumana Hatui

(Joint work with Manoj K. Yadav and L.R.Vermani)

Harish-Chandra Research Institute (HRI), India

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University of Birmingham

Overview

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3. $\mathrm{H}_2(G, \mathbb{Z}) \cong (\mathrm{H}^2(G, \mathbb{C}^\times))^*$ is known as Schur multiplier of G.

Definition

Definition (Internal central product)

Let G be group. A group G is called internal central product of its two normal subgroups H and K amalgamating A if $G = HK$ with $A = H \cap K$ and $[H, K] = 1$.

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Definition (External central product)

Let H, K be two groups with isomorphic subgroups $A \leq Z(H)$, $B \leq Z(K)$ under an isomorphism $\phi : A \to B$. Consider the normal subgroup $U = \{(a, \phi(a)^{-1}) \mid a \in A\}$. Then the group $G := (H \times K)/U$ is called the external central product of H and K amalgamating A and B via ϕ.

1. (I. Schur, 1907)
   $\mathrm{H}^2(H \times K, D) \cong \mathrm{H}^2(H, D) \times \mathrm{H}^2(K, D) \times \mathrm{Hom}(H \otimes K, D)$.

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2. (K.Tahara, 1972)
   If G is the semidirect product of a normal subgroup H and a subgroup K, then $\mathrm{H}^2(G, D) \cong \mathrm{H}^2(K, D) \times \widehat{\mathrm{H}^2}(G, D)$, where $\widehat{\mathrm{H}^2}(G, D)$ is the kernel of res: $\mathrm{H}^2(G, D) \to \mathrm{H}^2(K, D)$.

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3. Let G be a central product of two groups H and K. We study $H^2(G, D)$, in terms of the second cohomology groups of certain quotients of H and K.

Theorem (Wiegold, 1971)

Let H, K be finite groups, let U, V be isomorphic central subgroups of H, K respectively, and let ϕ be an isomorphism from U onto V. Then the multiplicator of the central product G of H and K amalgamating U with V according to ϕ contains a subgroup isomorphic with $H/U \otimes K/V$.

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Theorem (Eckmann, Hilton and Stammbach, 1973)

Let W be central in $A = H \times K$ with quotient G. Let U and V be the images of W under the projection of A onto H and K respectively. Then $H/U \otimes K/V$ is a quotient of $H_2(G, \mathbb{Z})$.

Preliminaries

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2. Then we have the exact sequence,

$$0 \to \mathrm{Hom}(N \cap X', D) \stackrel{\text{tra}}{\to} \mathrm{H}^2(X/N, D) \stackrel{\text{inf}}{\to} \mathrm{H}^2(X, D)$$

$$\stackrel{\chi}{\to} \mathrm{H}^2(N, D) \oplus \mathrm{Hom}(X \otimes N, D),$$

1. tra : transgression homomorphism
2. inf : inflation homomorphism
3. res : restriction homomorphism
4. $\chi = (\text{res}, \psi)$, defined by Iwahori, Matsumoto, where $\psi(\xi)(\bar{x}, n) = f(x, n) - f(n, x)$ for $\bar{x} = xX' \in X/X'$ and $n \in N$, where $\xi \in \mathrm{H}^2(X, D)$ and f is a 2-cocycle representative of ξ.

1. Define a map

$$\theta' : \mathrm{H}^2(G, D) \to \mathrm{H}^2(H, D) \oplus \mathrm{H}^2(K, D) \oplus \mathrm{Hom}(H \otimes K, D)$$

$$\theta' = (\mathrm{res}, \mathrm{res}, \nu)$$

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2. $\nu : \mathrm{H}^2(G, D) \to \mathrm{Hom}(H \otimes K, D)$ is a homomorphism defined as follows: If $\xi \in \mathrm{H}^2(G, D)$ is represented by a 2-cocycle f, then $\nu(\xi)$ is the homomorphism $\bar{f} \in \mathrm{Hom}(H \otimes K, D)$ defined by

$$\bar{f}(\bar{h} \otimes \bar{k}) = f(h, k) - f(k, h),$$

where $\bar{h} = hH'$ and $\bar{k} = kK'$.

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3. θ' is indeed a homomorphism.

- We have the exact sequence

$$(H \otimes A) \oplus (K \otimes A) \xrightarrow{\mu} H \otimes K \xrightarrow{\lambda} H/A \otimes K/A \to 0.$$

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$$\begin{array}{ccccc} 0 & \longrightarrow & \mathrm{Hom}(H/A \otimes K/A, D) & \xrightarrow{\lambda^*} & \mathrm{Hom}(H \otimes K, D) \\ & & & & \downarrow \mu^* \\ & & & & \mathrm{Hom}(H \otimes A, D) \oplus \mathrm{Hom}(K \otimes A, D) \end{array}$$

1. $X_1 = \mathrm{H}^2(A, D) \oplus \mathrm{Hom}(H \otimes A, D)$
2. $X_2 = \mathrm{H}^2(A, D) \oplus \mathrm{Hom}(K \otimes A, D)$
3. $X_3 = \mathrm{Hom}(H \otimes A, D) \oplus \mathrm{Hom}(K \otimes A, D)$
4. $Y = \mathrm{H}^2(A, D) \oplus \mathrm{H}^2(A, D)$

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathrm{Hom}(A\cap G') & \xrightarrow{(\mathrm{res},\mathrm{res})} & \mathrm{Hom}(A\cap H') \oplus \mathrm{Hom}(A\cap K') \\
\downarrow{\scriptstyle \mathrm{tra}} & & \downarrow{\scriptstyle (\mathrm{tra},\mathrm{tra},0)} \\
\mathrm{H}^2(G/A) & \xrightarrow{\theta} & \mathrm{H}^2(H/A) \oplus \mathrm{H}^2(K/A) \oplus \mathrm{Hom}(H/A \otimes K/A) \\
\downarrow{\scriptstyle \mathrm{inf}} & & \downarrow{\scriptstyle (\mathrm{inf},\mathrm{inf},\lambda^*)} \\
\mathrm{H}^2(G) & \xrightarrow{\theta'} & \mathrm{H}^2(H) \oplus \mathrm{H}^2(K) \oplus \mathrm{Hom}(H\otimes K) \\
\downarrow{\scriptstyle (\mathrm{res},\psi)} & & \downarrow{\scriptstyle \big((\mathrm{res},\psi),(\mathrm{res},\psi),\mu^*\big)} \\
\mathrm{H}^2(A) \oplus \mathrm{Hom}(G\otimes A) & & X_1 \oplus X_2 \oplus X_3 \\
 & \searrow{\scriptstyle (\triangle,\alpha^*,\alpha^*)} & \downarrow{\scriptstyle \cong} \\
 & & Y \oplus X_3 \oplus X_3.
\end{array}
$$

Diagram 1

Results

Lemma

$\mathrm{Ker}(\theta') = \{\inf(\eta) \mid \eta \in \theta^{-1}\big(\,\mathrm{Im}(\mathrm{tra}, \mathrm{tra}, 0)\big)\}.$

1. Set $Z = H' \cap K'$, where X' denotes the commutator subgroup of a group X.

Results

Lemma

$\mathrm{Ker}(\theta') = \{\inf(\eta) \mid \eta \in \theta^{-1}\big(\mathrm{Im}(\mathrm{tra}, \mathrm{tra}, 0)\big)\}.$

1. Set $Z = H' \cap K'$, where X' denotes the commutator subgroup of a group X.
2. We have an exact sequence

$$0 \to H' \cap K' \xrightarrow{\alpha_1} (A \cap H') \oplus (A \cap K') \xrightarrow{\alpha_2} A \cap G' \to 0,$$

which induces an exact sequence

1. $$0 \rightarrow \mathrm{Hom}(A \cap G', D) \xrightarrow{\alpha_2^*} \mathrm{Hom}(A \cap H', D) \oplus \mathrm{Hom}(A \cap K', D) \xrightarrow{\alpha_1^*} \mathrm{Hom}(Z, D) \rightarrow 0,$$

   α_2^* is the homomorphism $(\mathrm{res}, \mathrm{res})$.

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$$0\to\mathrm{Hom}(A\cap G', D)\xrightarrow{\alpha_2^*}\mathrm{Hom}(A\cap H', D)\oplus\mathrm{Hom}(A\cap K', D)\xrightarrow{\alpha_1^*}\mathrm{Hom}(Z, D)\to 0,$$

α_2^* is the homomorphism $(\mathrm{res}, \mathrm{res})$.

2. Define $\chi(f) = \inf\circ\theta^{-1}\circ(\mathrm{tra}, \mathrm{tra}, 0)(g)$ such that $f = \alpha_1^*(g)$.

Theorem

The following sequence is exact:

$$0\to\mathrm{Hom}(Z, D)\xrightarrow{\chi}\mathrm{H}^2(G, D)\xrightarrow{\theta'}\mathrm{H}^2(H, D)\oplus\mathrm{H}^2(K, D)\oplus\mathrm{Hom}(H\otimes K, D).$$

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\begin{array}{ccc}
0 & & 0 \\
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\mathrm{Hom}(A \cap G') & \xrightarrow{(\mathrm{res},\mathrm{res})} & \mathrm{Hom}(A \cap H') \oplus \mathrm{Hom}(A \cap K') \\
\Big\downarrow{\scriptstyle \mathrm{tra}} & & \Big\downarrow{\scriptstyle (\mathrm{tra},\mathrm{tra},0)} \\
\mathrm{H}^2(G/A) & \xrightarrow{\theta} & \mathrm{H}^2(H/A) \oplus \mathrm{H}^2(K/A) \oplus \mathrm{Hom}(H/A \otimes K/A) \\
\Big\downarrow{\scriptstyle \mathrm{inf}} & & \Big\downarrow{\scriptstyle (\mathrm{inf},\mathrm{inf},\lambda^*)} \\
\mathrm{H}^2(G) & \xrightarrow{\theta'} & \mathrm{H}^2(H) \oplus \mathrm{H}^2(K) \oplus \mathrm{Hom}(H \otimes K) \\
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\mathrm{H}^2(A) \oplus \mathrm{Hom}(G \otimes A) & & X_1 \oplus X_2 \oplus X_3 \\
& \searrow{\scriptstyle (\triangle,\alpha^*,\alpha^*)} & \Big\downarrow{\scriptstyle \cong} \\
& & Y \oplus X_3 \oplus X_3.
\end{array}
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Diagram 1

Theorem A

Let B be a subgroup of G such that $B \leq Z$. Then

$$\mathrm{H}^2(G, D) \cong \mathrm{H}^2(G/B, D)/N,$$

where $N \cong \mathrm{Hom}(B, D)$.

Theorem A

Let B be a subgroup of G such that $B \leq Z$. Then

$$\mathrm{H}^2(G, D) \cong \mathrm{H}^2(G/B, D)/N,$$

where $N \cong \mathrm{Hom}(B, D)$.

Corollary (Blackburn, Evens, 1979)

Let G be an extra-special p-group of order p^{2n+1}, $n \geq 2$. Then $M(G)$ is an elementary abelian p-group of order p^{2n^2-n-1}.

1. $L \cong \text{Hom}\big((A \cap H')/Z, D\big)$
2. $M \cong \text{Hom}\big((A \cap K')/Z, D\big)$
3. Consider $\inf : \text{H}^2(G/A, D) \to \text{H}^2(G/Z, D)$. Then

   $$\text{Im}(\inf) \cong \text{H}^2(H/A, D)/L \oplus \text{H}^2(K/A, D)/M \oplus \text{Hom}(H/A \otimes K/A, D)$$

   embeds in $\text{H}^2(G/Z, D)$.

- $\mathrm{H}^2(G, D) \cong \mathrm{H}^2(G/Z, D)/N$, where $N \cong \mathrm{Hom}(Z, D)$.

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$\mathrm{Hom}(Z, D)$ *embeds in* $\mathrm{H}^2(H/A, D)/L \oplus \mathrm{H}^2(K/A, D)/M$.

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$\mathrm{Hom}(Z, D)$ *embeds in* $\mathrm{H}^2(H/A, D)/L \oplus \mathrm{H}^2(K/A, D)/M$.

Corollary

$\mathrm{Hom}(Z, D)$ *embeds in* $\mathrm{H}^2(H/Z, D) \oplus \mathrm{H}^2(K/Z, D)$.

Theorem B

Let L, M be defined as above and $N \cong \mathrm{Hom}(Z, D)$. Then the following statements hold true:

(i) $\big(\mathrm{H}^2(H/A, D)/L \oplus \mathrm{H}^2(K/A, D)/M\big)/N \oplus \mathrm{Hom}(H/A \otimes K/A, D)$ embeds in $\mathrm{H}^2(G, D)$.

(ii) $\mathrm{H}^2(G, D)$ embeds in $\big(\mathrm{H}^2(H/Z, D) \oplus \mathrm{H}^2(K/Z, D)\big)/N \oplus \mathrm{Hom}(H \otimes K, D)$.

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2. $\mathrm{Im}(\mathrm{inf}) \cong \mathrm{H}^2(H/A, D)/L \oplus \mathrm{H}^2(K/A, D)/M \oplus \mathrm{Hom}(H/A \otimes K/A, D)$ embeds in $\mathrm{H}^2(G/Z, D)$.

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2. $\mathrm{Im}(\inf) \cong \mathrm{H}^2(H/A, D)/L \oplus \mathrm{H}^2(K/A, D)/M \oplus \mathrm{Hom}(H/A \otimes K/A, D)$ embeds in $\mathrm{H}^2(G/Z, D)$.
3. $\mathrm{H}^2(G/Z, D)$ embeds in $\mathrm{H}^2(H/Z, D)/L \oplus \mathrm{H}^2(K/Z, D)/M \oplus \mathrm{Hom}(H \otimes K, D)$.

Examples

- Neither of the two embeddings of Theorem B is an isomorphism.

1. **Example1:** Let H be the extraspecial p-groups of order p^3 and exponent p and $K = \mathbb{Z}_p^{(n+1)}$, where $n \geq 1$. Let G be a central product of H and K amalgamated at $A \cong H' \cong \mathbb{Z}_p$. Note that $G = H \times \mathbb{Z}_p^{(n)}$. It is easy to see that

$$M(G) \cong \mathbb{Z}_p^{\left(\frac{1}{2}n(n+3)+2\right)}.$$

• First embedding in Theorem B can very well be an isomorphism, but the second one can still be strict (i.e., not an isomorphism).

1. **Example 2.** Consider the group G presented as

$$G = \langle \alpha, \alpha_1, \alpha_2, \gamma \mid [\alpha_1, \alpha] = \gamma^{p^2} = \alpha_2, \alpha^p = \alpha_1^p = \alpha_2^p = 1 \rangle.$$

2. **Example 3.** Consider the group G presented as $G = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \gamma \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \gamma^p = \alpha_3, \alpha^p = \alpha_i^{(p)} = 1, i = 1, 2, 3 \rangle$.

- Both the embeddings in Theorem B can be isomorphisms.

1. **Example 4.** Let H be the extraspecial p-groups of order p^3 and exponent p^2 and $K \cong \mathbb{Z}_{p^{n+1}}$, the cyclic group of order p^{n+1}, where $n \geq 1$. Let G be a central product of H and K amalgamated at $A \cong H' \cong \mathbb{Z}_p$.

Theorem

If the second embedding in Theorem B is an isomorphism, then so is the first.

Corollary

(i) If $A = Z$*, then*

$$\mathrm{H}^2(G,D) \cong \big(\mathrm{H}^2(H/Z,D) \oplus \mathrm{H}^2(K/Z,D)\big) / \mathrm{Hom}(Z,D) \oplus \mathrm{Hom}(H/Z \otimes K/Z, D)$$

(ii) If $\inf : \mathrm{H}^2(H/A,D) \to \mathrm{H}^2(H/Z,D)$ *and* $\inf : \mathrm{H}^2(K/A,D) \to \mathrm{H}^2(K/Z,D)$ *are epimorphisms, then*

$$\mathrm{H}^2(G,D) \cong \big(\mathrm{H}^2(H/Z,D) \oplus \mathrm{H}^2(K/Z,D)\big) / \mathrm{Hom}(Z,D) \oplus \mathrm{Hom}(H/A \otimes K/A, D)$$

More precisely, the first embedding in Theorem B is an isomorphism.

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Thank you for your attention!