Fusion systems containing pearls

Valentina Grazian

University of Aberdeen

Groups St Andrews 2017
Birmingham, 7th August 2017

Fusion

Let G be a finite group. Two subgroups $P, Q \leq G$ are *fused* in G if there exists an element $g \in G$ such that $P^g = Q$.

Let G be a finite group. Two subgroups $P, Q \leq G$ are *fused* in G if there exists an element $g \in G$ such that $P^g = Q$.

Definition

Let p be a prime and let S be a Sylow p-subgroup of G. The fusion category of G on S is the category $\mathcal{F}_S(G)$ whose objects are the subgroups of S and whose morphism sets are:

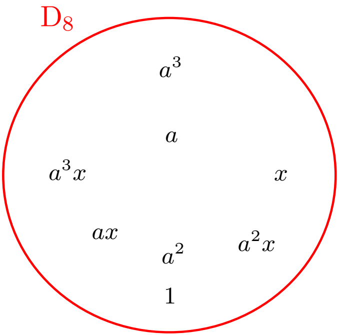
$$\mathrm{Mor}_{\mathcal{F}_S(G)}(P,Q) = \mathrm{Hom}_G(P,Q) = \{c_g|_P \colon P \to Q | g \in G, P^g \leq Q\},$$

for every $P, Q \leq S$.

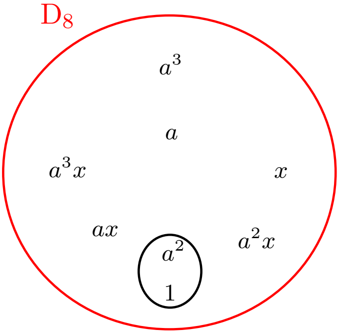
Pick a p-group S.

$S = \mathrm{D}_8$

$\mathrm{D}_8 := \langle a, x \mid a^4 = x^2 = 1, a^x = a^3 \rangle$



Pick a p-group S. Pick a finite group G such that $S \in \mathrm{Syl}_p(G)$.

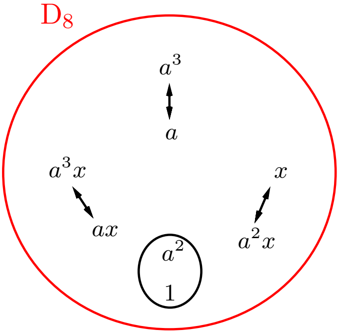


$$S = \mathrm{D}_8$$

$$\mathrm{D}_8 := \langle a, x \mid a^4 = x^2 = 1, a^x = a^3 \rangle$$

$$G = \mathrm{D}_8$$

Pick a p-group S. Pick a finite group G such that $S \in \mathrm{Syl}_p(G)$.



$$S = D_8$$

$$D_8 := \langle a, x \mid a^4 = x^2 = 1, a^x = a^3 \rangle$$

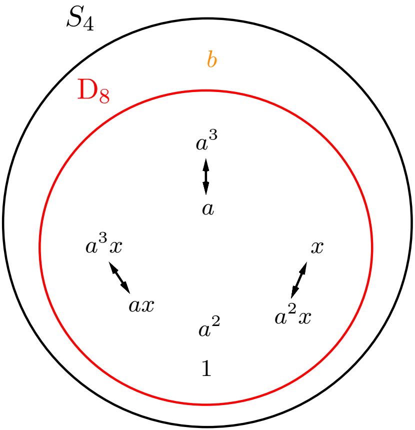
$$G = D_8$$

Consider the conjugation maps by elements $g \in G$ that *fuse* some elements/subgroups of S.

fusion is determined by $\mathrm{Inn}(D_8)$:

$\mathcal{F}_{D_8}(D_8) = \langle \mathrm{Inn}(D_8) \rangle$

Pick a p-group S. Pick a finite group G such that $S \in \mathrm{Syl}_p(G)$.



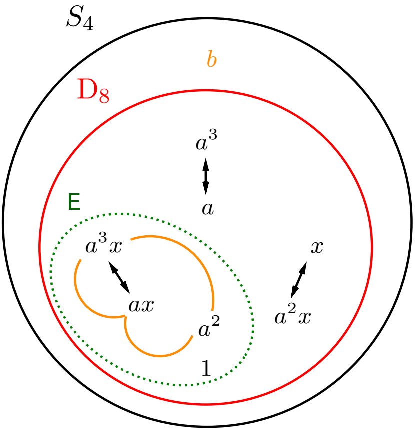
$S = D_8$

$D_8 := \langle a, x \mid a^4 = x^2 = 1, a^x = a^3 \rangle$

$G = \mathrm{Sym}(4) = S_4$

$a = (1234),\ x = (13).$

Pick a p-group S. Pick a finite group G such that $S \in \mathrm{Syl}_p(G)$.



$$S = \mathrm{D}_8$$

$$\mathrm{D}_8 := \langle a, x \mid a^4 = x^2 = 1, a^x = a^3 \rangle$$

$$G = \mathrm{Sym}(4) = S_4$$

$a = (1234),\ x = (13).$

$b = (123)$

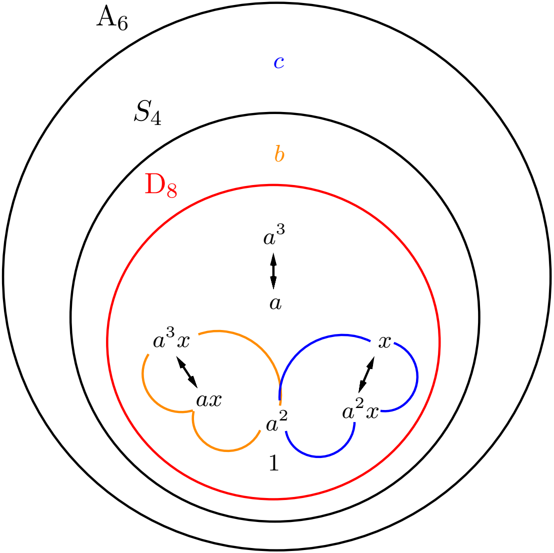
$$(a^2)^b = ((13)(24))^{(123)} = (12)(34) = ax$$

fusion is determined by

$\mathrm{Inn}(\mathrm{D}_8)$ and $\mathrm{Aut}(E) \cong \mathrm{SL}_2(2) \cong S_3$

$\mathcal{F}_{\mathrm{D}_8}(S_4) = \langle \mathrm{Inn}(\mathrm{D}_8), \mathrm{Aut}(E) \rangle$

Pick a p-group S. Pick a finite group G such that $S \in \mathrm{Syl}_p(G)$.



$S = D_8$

$D_8 := \langle a, x \mid a^4 = x^2 = 1, a^x = a^3 \rangle$

$G = A_6$

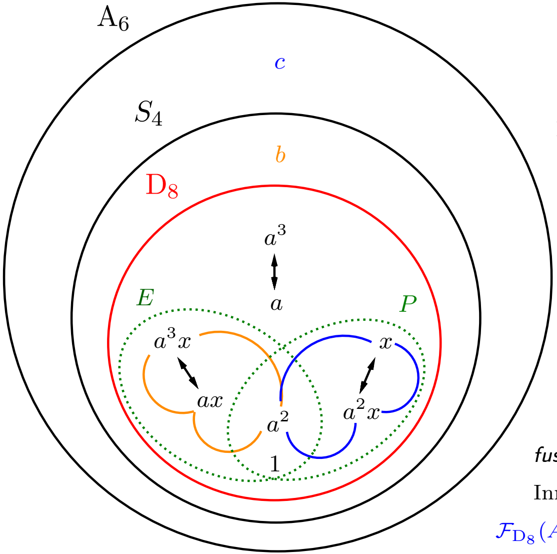
$a = (1234)(56), x = (13)(56).$

$c = (25)(46)$

$(a^2)^c = ((13)(24))^{(25)(46)}$

$= (13)(56) = x$

Pick a p-group S. Pick a finite group G such that $S \in \mathrm{Syl}_p(G)$.



$S = \mathrm{D}_8$

$\mathrm{D}_8 := \langle a, x \mid a^4 = x^2 = 1, a^x = a^3 \rangle$

$G = A_6$

$a = (1234)(56), x = (13)(56).$

$c = (25)(46)$

$(a^2)^c = ((13)(24))^{(25)(46)}$

$= (13)(56) = x$

fusion is determined by

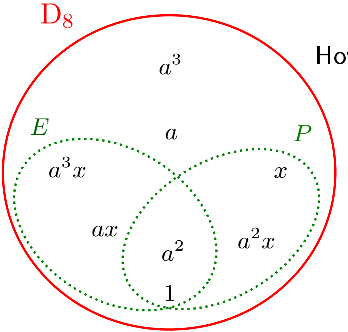
$\mathrm{Inn}(\mathrm{D}_8)$, $\mathrm{Aut}(E)$ and $\mathrm{Aut}(P)$

$\mathcal{F}_{\mathrm{D}_8}(A_6) = \langle \mathrm{Inn}(\mathrm{D}_8), \mathrm{Aut}(E), \mathrm{Aut}(P) \rangle$

Pick a p-group S.

$$S = \mathrm{D}_8$$

$$\mathrm{D}_8 := \langle a, x \mid a^4 = x^2 = 1, a^x = a^3 \rangle$$



How many ways to fuse elements of D_8?

– $\mathcal{F}_{\mathrm{D}_8}(\mathrm{D}_8)$ (no essential subgroups)

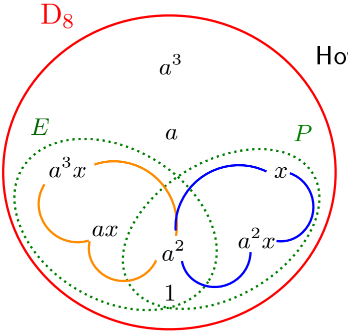
– $\mathcal{F}_{\mathrm{D}_8}(S_4)$ (E essential)

– $\mathcal{F}_{\mathrm{D}_8}(A_6)$ (E and P essential)

Pick a p-group S.

$$S = \mathrm{D}_8$$

$$\mathrm{D}_8 := \langle a, x \mid a^4 = x^2 = 1, a^x = a^3 \rangle$$



How many ways to fuse elements of D_8?

– $\mathcal{F}_{\mathrm{D}_8}(\mathrm{D}_8)$ (no essential subgroups)

– $\mathcal{F}_{\mathrm{D}_8}(S_4)$ (E essential)

– $\mathcal{F}_{\mathrm{D}_8}(A_6)$ (E and P essential)

Let p be a prime and let S be a p-group. Choose a collection of morphisms between subgroups of S that "behave" as conjugation maps.

Definition (Fusion System)

A Fusion system $\mathcal{F}$ on S is a category whose objects are the subgroups of S and with morphism sets $\mathrm{Hom}_{\mathcal{F}}(P, Q) \subseteq \mathrm{Inj}(P, Q)$ such that

1. $\mathrm{Hom}_S(P, Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P, Q)$,
2. each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ is the composition of an isomorphism $\alpha \in \mathrm{Mor}(\mathcal{F})$ and an inclusion $\beta \in \mathrm{Mor}(\mathcal{F})$.

Let p be a prime and let S be a p-group. Choose a collection of morphisms between subgroups of S that "behave" as conjugation maps.

Definition (Fusion System)

A Fusion system $\mathcal{F}$ on S is a category whose objects are the subgroups of S and with morphism sets $\mathrm{Hom}_{\mathcal{F}}(P,Q) \subseteq \mathrm{Inj}(P,Q)$ such that

1. $\mathrm{Hom}_S(P,Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P,Q)$,
2. each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ is the composition of an isomorphism $\alpha \in \mathrm{Mor}(\mathcal{F})$ and an inclusion $\beta \in \mathrm{Mor}(\mathcal{F})$.

A Fusion system is Saturated if it satisfies certain extra properties (Sylow and extension properties).

Let p be a prime and let S be a p-group. Choose a collection of morphisms between subgroups of S that "behave" as conjugation maps.

Definition (Fusion System)

A Fusion system $\mathcal{F}$ on S is a category whose objects are the subgroups of S and with morphism sets $\mathrm{Hom}_{\mathcal{F}}(P, Q) \subseteq \mathrm{Inj}(P, Q)$ such that

1. $\mathrm{Hom}_S(P, Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P, Q)$,
2. each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ is the composition of an isomorphism $\alpha \in \mathrm{Mor}(\mathcal{F})$ and an inclusion $\beta \in \mathrm{Mor}(\mathcal{F})$.

A Fusion system is Saturated if it satisfies certain extra properties (Sylow and extension properties).

If $S \in \mathrm{Syl}_p(G)$ then the fusion category $\mathcal{F}_S(G)$ is a saturated fusion system on S.

Let p be a prime and let S be a p-group. Choose a collection of morphisms between subgroups of S that "behave" as conjugation maps.

Definition (Fusion System)

A **Fusion system** $\mathcal{F}$ on S is a category whose objects are the subgroups of S and with morphism sets $\mathrm{Hom}_{\mathcal{F}}(P,Q) \subseteq \mathrm{Inj}(P,Q)$ such that

1. $\mathrm{Hom}_S(P,Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P,Q)$,
2. each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ is the composition of an isomorphism $\alpha \in \mathrm{Mor}(\mathcal{F})$ and an inclusion $\beta \in \mathrm{Mor}(\mathcal{F})$.

A Fusion system is **Saturated** if it satisfies certain extra properties (Sylow and extension properties).

If $S \in \mathrm{Syl}_p(G)$ then the fusion category $\mathcal{F}_S(G)$ is a saturated fusion system on S.

If $\mathcal{F}$ is a saturated fusion system on S and there is no finite group G such that $S \in \mathrm{Syl}_p(G)$ and $\mathcal{F} = \mathcal{F}_S(G)$, then $\mathcal{F}$ is called **exotic**.

Let p be a prime and let S be a p-group. Choose a collection of morphisms between subgroups of S that "behave" as conjugation maps.

Definition (Fusion System)

A Fusion system $\mathcal{F}$ on S is a category whose objects are the subgroups of S and with morphism sets $\mathrm{Hom}_{\mathcal{F}}(P, Q) \subseteq \mathrm{Inj}(P, Q)$ such that

1. $\mathrm{Hom}_S(P, Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P, Q)$,
2. each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ is the composition of an isomorphism $\alpha \in \mathrm{Mor}(\mathcal{F})$ and an inclusion $\beta \in \mathrm{Mor}(\mathcal{F})$.

A Fusion system is Saturated if it satisfies certain extra properties (Sylow and extension properties).

If $S \in \mathrm{Syl}_p(G)$ then the fusion category $\mathcal{F}_S(G)$ is a saturated fusion system on S.

If $\mathcal{F}$ is a saturated fusion system on S and there is no finite group G such that $S \in \mathrm{Syl}_p(G)$ and $\mathcal{F} = \mathcal{F}_S(G)$, then $\mathcal{F}$ is called exotic.

Question (suggested by Oliver)

Try to better understand how exotic fusion systems arise at odd primes.

Question

Given a class of p-groups, can we determine all the *simple* fusion systems on them?

Question

Given a class of p-groups, can we determine all the *simple* fusion systems on them?

An interesting class to study is the class of p-groups having small sectional rank.

Question

Given a class of p-groups, can we determine all the *simple* fusion systems on them?

An interesting class to study is the class of p-groups having small sectional rank.

A p-group S has sectional rank k if every elementary abelian section P/Q of S has order at most p^k (equivalently if every subgroup of S can be generated by k elements).

Question

Given a class of p-groups, can we determine all the *simple* fusion systems on them?

An interesting class to study is the class of p-groups having small sectional rank.

A p-group S has sectional rank k if every elementary abelian section P/Q of S has order at most p^k (equivalently if every subgroup of S can be generated by k elements).

- Oliver, 2016 (Memoir of the AMS): $p = 2$, sectional rank at most 4;

Question

Given a class of p-groups, can we determine all the *simple* fusion systems on them?

An interesting class to study is the class of p-groups having small sectional rank.

A p-group S has sectional rank k if every elementary abelian section P/Q of S has order at most p^k (equivalently if every subgroup of S can be generated by k elements).

- Oliver, 2016 (Memoir of the AMS): $p = 2$, sectional rank at most 4;
- Diaz, Ruiz and Viruel, 2007: p odd, sectional rank 2;

Question

Given a class of p-groups, can we determine all the *simple* fusion systems on them?

An interesting class to study is the class of p-groups having small sectional rank.

A p-group S has sectional rank k if every elementary abelian section P/Q of S has order at most p^k (equivalently if every subgroup of S can be generated by k elements).

- Oliver, 2016 (Memoir of the AMS): $p = 2$, sectional rank at most 4;
- Diaz, Ruiz and Viruel, 2007: p odd, sectional rank 2;
- G., 2017 (PhD thesis): $p \geq 5$, sectional rank 3.

Question

Given a class of p-groups, can we determine all the *simple* fusion systems on them?

An interesting class to study is the class of p-groups having small sectional rank.

A p-group S has sectional rank k if every elementary abelian section P/Q of S has order at most p^k (equivalently if every subgroup of S can be generated by k elements).

- Oliver, 2016 (Memoir of the AMS): $p = 2$, sectional rank at most 4;
- Diaz, Ruiz and Viruel, 2007: p odd, sectional rank 2;
- G., 2017 (PhD thesis): $p \geq 5$, sectional rank 3.

Question

Given a class of p-groups, can we determine all the *simple* fusion systems on them?

An interesting class to study is the class of p-groups having small sectional rank.

A p-group S has sectional rank k if every elementary abelian section P/Q of S has order at most p^k (equivalently if every subgroup of S can be generated by k elements).

- Oliver, 2016 (Memoir of the AMS): $p = 2$, sectional rank at most 4;
- Diaz, Ruiz and Viruel, 2007: p odd, sectional rank 2;
- G., 2017 (PhD thesis): $p \geq 5$, sectional rank 3.

Theorem 1 (G., 2016)

Let $p \geq 5$ be a prime and let $\mathcal{F}$ be a saturated fusion system on the p-group S. If S has sectional rank 3 and $O_p(\mathcal{F}) = 1$ then there exists an essential subgroup E of S such that either $E \cong \mathrm{C}_p \times \mathrm{C}_p$ or $E \cong p_+^{1+2}$.

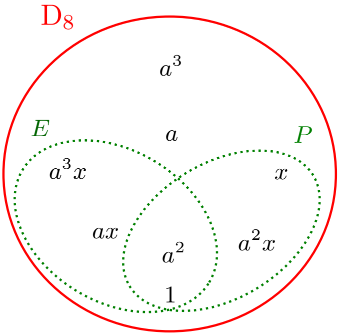
Theorem 1 (G., 2016)

Let $p \geq 5$ be a prime and let $\mathcal{F}$ be a saturated fusion system on the p-group S. If S has sectional rank 3 and $O_p(\mathcal{F}) = 1$ then there exists an essential subgroup E of S such that either $E \cong \mathrm{C}_p \times \mathrm{C}_p$ or $E \cong p_+^{1+2}$.

Theorem 1 (G., 2016)

Let $p \geq 5$ be a prime and let $\mathcal{F}$ be a saturated fusion system on the p-group S. If S has sectional rank 3 and $O_p(\mathcal{F}) = 1$ then there exists an essential subgroup E of S such that either $E \cong \mathrm{C}_p \times \mathrm{C}_p$ or $E \cong p_+^{1+2}$.

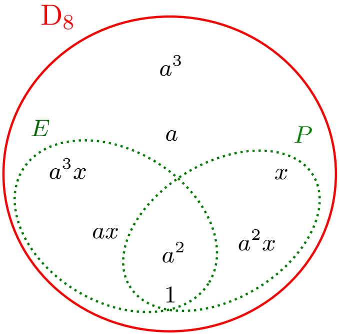
Let $S = \mathrm{D}_8$. E and P are the only subgroups of S that can be essential and $E \cong P \cong \mathrm{C}_2 \times \mathrm{C}_2$.



Theorem 1 (G., 2016)

Let $p \geq 5$ be a prime and let $\mathcal{F}$ be a saturated fusion system on the p-group S. If S has sectional rank 3 and $O_p(\mathcal{F}) = 1$ then there exists an essential subgroup E of S such that either $E \cong \mathrm{C}_p \times \mathrm{C}_p$ or $E \cong p_+^{1+2}$.

Let $S = \mathrm{D}_8$. E and P are the only subgroups of S that can be essential and $E \cong P \cong \mathrm{C}_2 \times \mathrm{C}_2$.



Can we characterize saturated fusion systems containing an essential subgroup that is either elementary abelian of order p^2 or non-abelian of order p^3 and exponent p?

Pearls

Let $\mathcal{F}$ be a saturated fusion system on the p-group S.

A pearl is an essential subgroup E of S that is either elementary abelian of order p^2 ($E \cong \mathrm{C}_p \times \mathrm{C}_p$) or non-abelian of order p^3 and exponent p (if p is odd then $E \cong p_+^{1+2}$).

Pearls

Let $\mathcal{F}$ be a saturated fusion system on the p-group S.

A pearl is an essential subgroup E of S that is either elementary abelian of order p^2 ($E \cong \mathrm{C}_p \times \mathrm{C}_p$) or non-abelian of order p^3 and exponent p (if p is odd then $E \cong p_+^{1+2}$).

Property 1: Essential subgroups of S are self-centralizing in S.

Pearls

Let $\mathcal{F}$ be a saturated fusion system on the p-group S.

A pearl is an essential subgroup E of S that is either elementary abelian of order p^2 ($E \cong \mathrm{C}_p \times \mathrm{C}_p$) or non-abelian of order p^3 and exponent p (if p is odd then $E \cong p_+^{1+2}$).

Property 1: Essential subgroups of S are self-centralizing in S.

Theorem (Suzuki): If a p-group S contains a subgroup E of order p^2 such that $\mathrm{C}_S(E) = E$ then S has maximal nilpotency class.

Pearls

Let $\mathcal{F}$ be a saturated fusion system on the p-group S.

A pearl is an essential subgroup E of S that is either elementary abelian of order p^2 ($E \cong \mathrm{C}_p \times \mathrm{C}_p$) or non-abelian of order p^3 and exponent p (if p is odd then $E \cong p_+^{1+2}$).

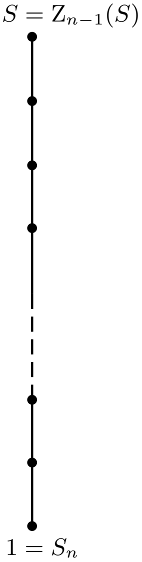
Property 1: Essential subgroups of S are self-centralizing in S.

Theorem (Suzuki): If a p-group S contains a subgroup E of order p^2 such that $\mathrm{C}_S(E) = E$ then S has maximal nilpotency class.

Theorem

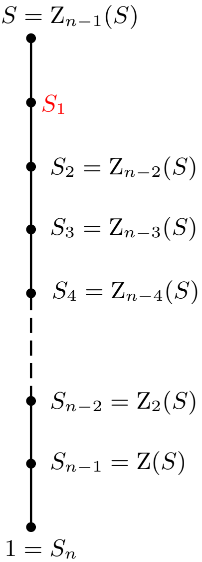
If $\mathcal{F}$ contains a pearl then S has maximal nilpotency class.

p-groups having maximal nilpotency class



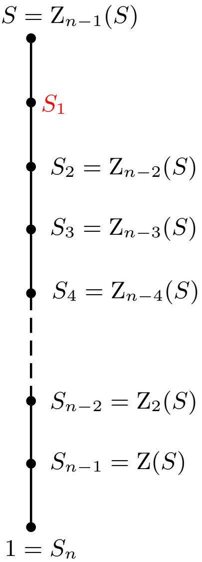
Let S be a p-group having order p^n and maximal nilpotency class (i.e. class $n-1$).

p-groups having maximal nilpotency class



Let S be a p-group having order p^n and maximal nilpotency class (i.e. class $n-1$). Set $S_2 = [S, S]$, $S_i = [S_{i-1}, S]$ for every $i \geq 3$ and $S_1 = C_S(S_2/S_4)$.

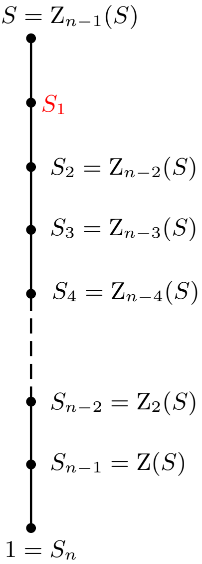
p-groups having maximal nilpotency class



Let S be a p-group having order p^n and maximal nilpotency class (i.e. class $n-1$). Set $S_2 = [S, S]$, $S_i = [S_{i-1}, S]$ for every $i \geq 3$ and $S_1 = C_S(S_2/S_4)$.

Properties of S_1:

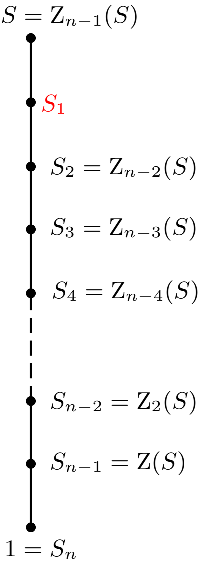
- $S_2 \leq S_1$;



Let S be a p-group having order p^n and maximal nilpotency class (i.e. class $n-1$). Set $S_2 = [S,S]$, $S_i = [S_{i-1}, S]$ for every $i \geq 3$ and $S_1 = C_S(S_2/S_4)$.

Properties of S_1:

- $S_2 \leq S_1$;
- $[S \colon S_1] = [S_1 \colon S_2] = p$;

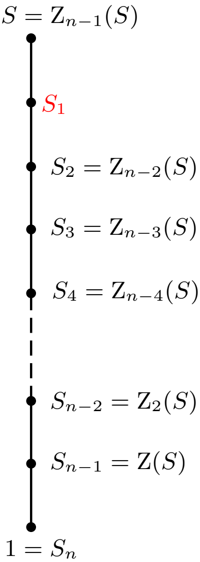


Let S be a p-group having order p^n and maximal nilpotency class (i.e. class $n-1$). Set $S_2 = [S, S]$, $S_i = [S_{i-1}, S]$ for every $i \geq 3$ and $S_1 = C_S(S_2/S_4)$.

Properties of S_1:

- $S_2 \leq S_1$;
- $[S\colon S_1] = [S_1\colon S_2] = p$;
- S_1 is characteristic in S; and

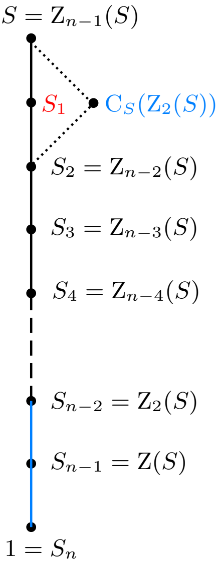
p-groups having maximal nilpotency class



Let S be a p-group having order p^n and maximal nilpotency class (i.e. class $n-1$). Set $S_2 = [S, S]$, $S_i = [S_{i-1}, S]$ for every $i \geq 3$ and $S_1 = \mathrm{C}_S(S_2/S_4)$.

Properties of S_1:

- $S_2 \leq S_1$;
- $[S\colon S_1] = [S_1\colon S_2] = p$;
- S_1 is characteristic in S; and
- $S_1 = \mathrm{C}_S(S_i/S_{i+2})$ for every $2 \leq i \leq n-3$.

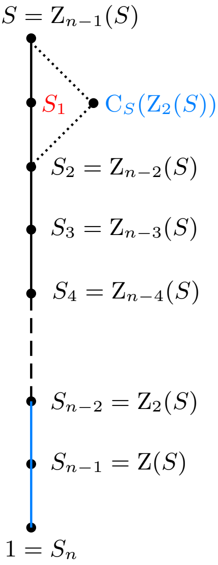


Let S be a p-group having order p^n and maximal nilpotency class (i.e. class $n-1$). Set $S_2 = [S, S]$, $S_i = [S_{i-1}, S]$ for every $i \geq 3$ and $S_1 = C_S(S_2/S_4)$.

Properties of S_1:

- $S_2 \leq S_1$;
- $[S\colon S_1] = [S_1\colon S_2] = p$;
- S_1 is characteristic in S; and
- $S_1 = C_S(S_i/S_{i+2})$ for every $2 \leq i \leq n-3$.

p-groups having maximal nilpotency class



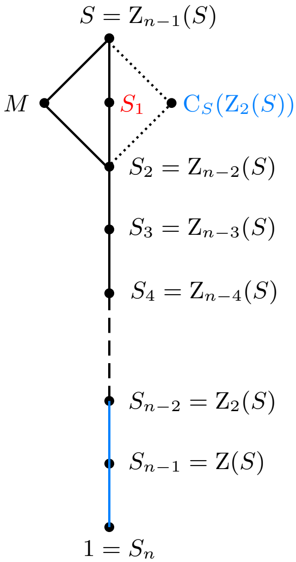
Let S be a p-group having order p^n and maximal nilpotency class (i.e. class $n-1$). Set $S_2 = [S, S]$, $S_i = [S_{i-1}, S]$ for every $i \geq 3$ and $S_1 = C_S(S_2/S_4)$.

Properties of S_1:

- $S_2 \leq S_1$;
- $[S\colon S_1] = [S_1\colon S_2] = p$;
- S_1 is characteristic in S; and
- $S_1 = C_S(S_i/S_{i+2})$ for every $2 \leq i \leq n-3$.

Another maximal subgroup of S is the group $C_S(S_{n-2}) = C_S(Z_2(S))$, that might coincide with S_1.

p-groups having maximal nilpotency class



Let S be a p-group having order p^n and maximal nilpotency class (i.e. class $n-1$). Set $S_2 = [S, S]$, $S_i = [S_{i-1}, S]$ for every $i \geq 3$ and $S_1 = C_S(S_2/S_4)$.

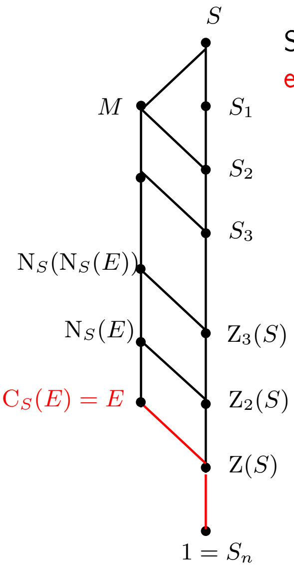
Properties of S_1:

- $S_2 \leq S_1$;
- $[S\colon S_1] = [S_1\colon S_2] = p$;
- S_1 is characteristic in S; and
- $S_1 = C_S(S_i/S_{i+2})$ for every $2 \leq i \leq n-3$.

Another maximal subgroup of S is the group $C_S(S_{n-2}) = C_S(Z_2(S))$, that might coincide with S_1.

Every other maximal subgroup of S has maximal nilpotency class.

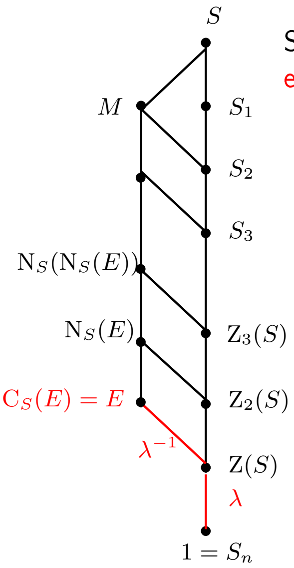
Structure of a p-group containing an abelian pearl



Suppose p is odd and $E \cong C_p \times C_p$ is an essential subgroup of the p-group S. Then:

- Property 1: $C_S(E) = E$;

Structure of a p-group containing an abelian pearl



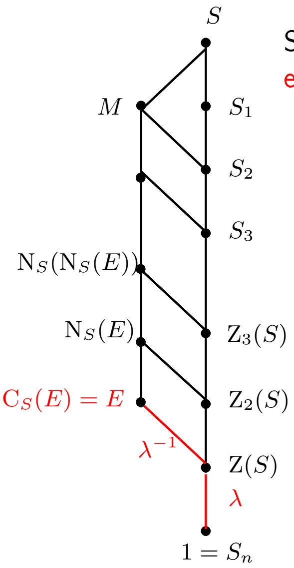
Suppose p is odd and $E \cong C_p \times C_p$ is an essential subgroup of the p-group S. Then:

- Property 1: $C_S(E) = E$;
- Property 2: there exists a non-trivial automorphism φ of S ($\varphi \in \mathrm{Aut}_{\mathcal{F}}(S)$) normalizing E such that

$$\varphi|_E = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix},$$

for some $\lambda \in \mathrm{GF}(p)$ having order $p - 1$.

Structure of a p-group containing an abelian pearl



Suppose p is odd and $E \cong C_p \times C_p$ is an essential subgroup of the p-group S. Then:

- Property 1: $C_S(E) = E$;
- Property 2: there exists a non-trivial automorphism φ of S ($\varphi \in \mathrm{Aut}_{\mathcal{F}}(S)$) normalizing E such that

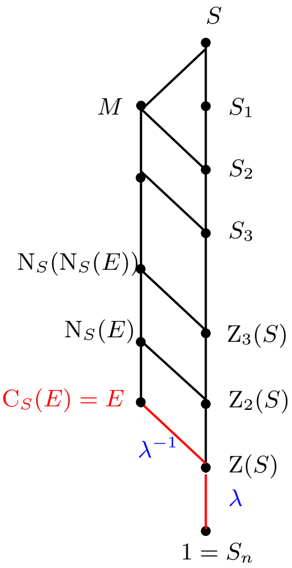
$$\varphi|_E = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix},$$

for some $\lambda \in \mathrm{GF}(p)$ having order $p-1$.

So if $E = \langle e \rangle \times \langle z \rangle$, with $z \in Z(S)$, then

$$e\varphi = e^{\lambda^{-1}} \text{ and } z\varphi = z^{\lambda}.$$

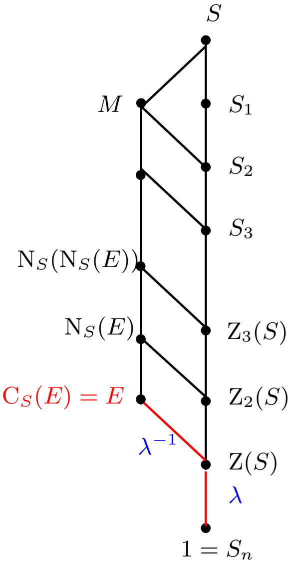
Structure of a p-group containing an abelian pearl



We can prove that

$$E \not\leq S_1 \text{ and } E \not\leq C_S(Z_2(S)).$$

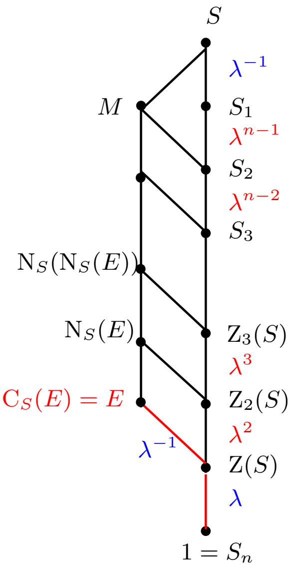
Structure of a p-group containing an abelian pearl



We can prove that

$$E \not\leq S_1 \text{ and } E \not\leq C_S(Z_2(S)).$$

In particular, recalling that $S_1 = C_S(S_i/S_{i+2})$ for every $2 \leq i \leq n-3$, we get that $[E, S_i] \not\leq S_{i+2}$ for every $i \geq 1$.



We can prove that

$$E \not\leq S_1 \text{ and } E \not\leq \mathrm{C}_S(\mathrm{Z}_2(S)).$$

In particular, recalling that $S_1 = \mathrm{C}_S(S_i/S_{i+2})$ for every $2 \leq i \leq n-3$, we get that $[E, S_i] \not\leq S_{i+2}$ for every $i \geq 1$.

This fact and properties of commutators enable us to determine the action of φ on every quotient S_i/S_{i+1}.

Main result

Let p be an odd prime and let $\mathcal{F}$ be a saturated fusion system on the p-group S containing a pearl.

Theorem 2 (G., 2017)

Suppose that S has sectional rank k and order $|S| \geq p^4$.

Main result

Let p be an odd prime and let $\mathcal{F}$ be a saturated fusion system on the p-group S containing a pearl.

Theorem 2 (G., 2017)

Suppose that S has sectional rank k and order $|S| \geq p^4$. Then $p \geq k \geq 2$ and one of the following holds:

- $|S| = p^{k+1}$, S_1 is elementary abelian and $\mathcal{F}$, if reduced, is known (Craven, Oliver, Semeraro);

Main result

Let p be an odd prime and let $\mathcal{F}$ be a saturated fusion system on the p-group S containing a pearl.

Theorem 2 (G., 2017)

Suppose that S has sectional rank k and order $|S| \geq p^4$. Then $p \geq k \geq 2$ and one of the following holds:

- $|S| = p^{k+1}$, S_1 is elementary abelian and $\mathcal{F}$, if reduced, is known (Craven, Oliver, Semeraro);
- $k = p - 1$ and $|S| \geq p^{p+1}$;

Main result

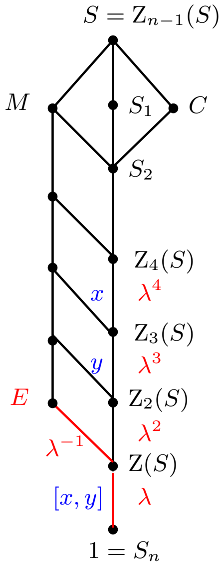
Let p be an odd prime and let $\mathcal{F}$ be a saturated fusion system on the p-group S containing a pearl.

Theorem 2 (G., 2017)

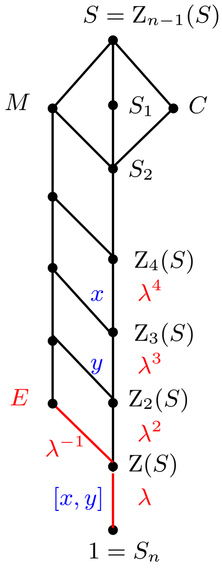
Suppose that S has sectional rank k and order $|S| \geq p^4$. Then $p \geq k \geq 2$ and one of the following holds:

- $|S| = p^{k+1}$, S_1 is elementary abelian and $\mathcal{F}$, if reduced, is known (Craven, Oliver, Semeraro);
- $k = p - 1$ and $|S| \geq p^{p+1}$;
- $k \geq 3$, $k + 3 \leq p \leq 2k + 1$, S has exponent p and $|S| \leq p^{p-1}$.

Methods



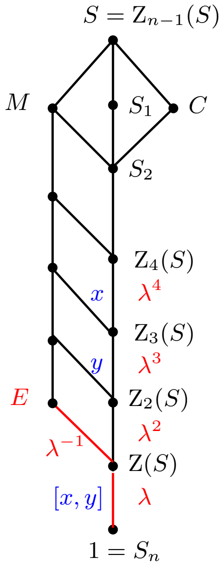
Note that the group $Z_3(S)$ is always abelian (it centralizes $Z_2(S)$).



Note that the group $Z_3(S)$ is always abelian (it centralizes $Z_2(S)$).

Suppose $Z_4(S)$ is NOT abelian.

Methods

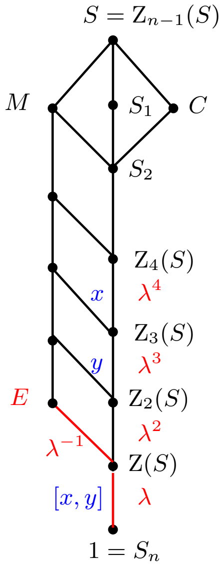


Note that the group $Z_3(S)$ is always abelian (it centralizes $Z_2(S)$).

Suppose $Z_4(S)$ is NOT abelian.

Then $|S| \leq p^7$ and $[Z_4(S), Z_3(S)] = Z(S)$.

Methods



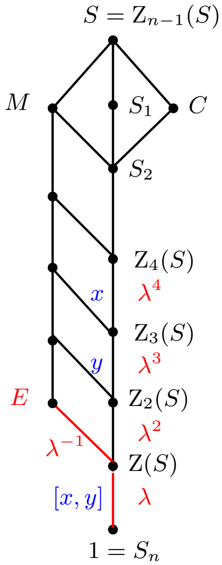
Note that the group $Z_3(S)$ is always abelian (it centralizes $Z_2(S)$).

Suppose $Z_4(S)$ is NOT abelian.

Then $|S| \le p^7$ and $[Z_4(S), Z_3(S)] = Z(S)$.

So there exists $x \in Z_4(S)$ and $y \in Z_3(S)$ such that $1 \neq [x,y] \in Z(S)$.

Methods



Note that the group $Z_3(S)$ is always abelian (it centralizes $Z_2(S)$).

Suppose $Z_4(S)$ is NOT abelian.

Then $|S| \leq p^7$ and $[Z_4(S), Z_3(S)] = Z(S)$.

So there exists $x \in Z_4(S)$ and $y \in Z_3(S)$ such that $1 \neq [x,y] \in Z(S)$. Thus

$$[x,y]^{\lambda} = [x,y]\varphi = [x^{\lambda^4}, y^{\lambda^3}] = [x,y]^{\lambda^7}.$$

Since λ has order $p-1$, this implies

$$p = 3 \quad \text{or} \quad p = 7.$$

Application: fusion systems on p-groups of sectional rank 3

Let p be an odd prime and let $\mathcal{F}$ be a saturated fusion system on the p-group S containing a pearl.

Theorem 2 (G., 2017)

Suppose that S has sectional rank k and order $|S| \geq p^4$. Then $p \geq k \geq 2$ and one of the following holds:

- $|S| = p^{k+1}$ and S_1 is elementary abelian;
- $k = p - 1$ and $|S| \geq p^{p+1}$;
- $k \geq 3$, $k + 3 \leq p \leq 2k + 1$, S has exponent p and $|S| \leq p^{p-1}$.

Application: fusion systems on p-groups of sectional rank 3

Let p be an odd prime and let $\mathcal{F}$ be a saturated fusion system on the p-group S containing a pearl.

Theorem 2 (G., 2017)

Suppose that S has sectional rank k and order $|S| \geq p^4$. Then $p \geq k \geq 2$ and one of the following holds:

- $|S| = p^{k+1}$ and S_1 is elementary abelian;
- $k = p-1$ and $|S| \geq p^{p+1}$;
- $k \geq 3$, $k+3 \leq p \leq 2k+1$, S has exponent p and $|S| \leq p^{p-1}$.

Corollary

Suppose that S has sectional rank $k = 3$. Then one of the following holds:

- $|S| = p^4$ and $S \in \mathrm{Syl}_p(\mathrm{Sp}_4(p))$;

Application: fusion systems on p-groups of sectional rank 3

Let p be an odd prime and let $\mathcal{F}$ be a saturated fusion system on the p-group S containing a pearl.

Theorem 2 (G., 2017)

Suppose that S has sectional rank k and order $|S| \geq p^4$. Then $p \geq k \geq 2$ and one of the following holds:

- $|S| = p^{k+1}$ and S_1 is elementary abelian;
- $k = p - 1$ and $|S| \geq p^{p+1}$;
- $k \geq 3$, $k + 3 \leq p \leq 2k + 1$, S has exponent p and $|S| \leq p^{p-1}$.

Corollary

Suppose that S has sectional rank $k = 3$. Then one of the following holds:

- $|S| = p^4$ and $S \in \mathrm{Syl}_p(\mathrm{Sp}_4(p))$;
- $3 = p - 1$ (impossible);

Application: fusion systems on p-groups of sectional rank 3

Let p be an odd prime and let $\mathcal{F}$ be a saturated fusion system on the p-group S containing a pearl.

Theorem 2 (G., 2017)

Suppose that S has sectional rank k and order $|S| \geq p^4$. Then $p \geq k \geq 2$ and one of the following holds:

- $|S| = p^{k+1}$ and S_1 is elementary abelian;
- $k = p - 1$ and $|S| \geq p^{p+1}$;
- $k \geq 3$, $k + 3 \leq p \leq 2k + 1$, S has exponent p and $|S| \leq p^{p-1}$.

Corollary

Suppose that S has sectional rank $k = 3$. Then one of the following holds:

- $|S| = p^4$ and $S \in \mathrm{Syl}_p(\mathrm{Sp}_4(p))$;
- $3 = p - 1$ (impossible);
- $p = 7$ and $S \cong$`SmallGroup(7^5,37)` (has order 7^5 and exponent 7).

Application: fusion systems on p-groups of sectional rank 3

Theorem 3 (G., 2017)

Let $p \geq 5$ be a prime, let $\mathcal{F}$ be a saturated fusion system on the p-group S. Suppose that $O_p(\mathcal{F}) = 1$ and S has sectional rank 3.
Then $\mathcal{F}$ contains a pearl and so one of the following holds:

- $|S| = p^4$ and $S \in \mathrm{Syl}_p(\mathrm{Sp}_4(p))$;
- $p = 7$, $S \cong$`SmallGroup(7^5,37)` (has order 7^5 and exponent 7), $\mathcal{F} = \langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(E) \rangle$, where $E \cong \mathrm{C}_7 \times \mathrm{C}_7$ is an abelian pearl, and $\mathcal{F}$ is simple and exotic.

Thank you.