How many composition factors of order p are there in a completely reducible subgroup of $\mathrm{GL}(d, p^f)$?

(joint work with M. Giudici, C. H. Li and G. Verret)

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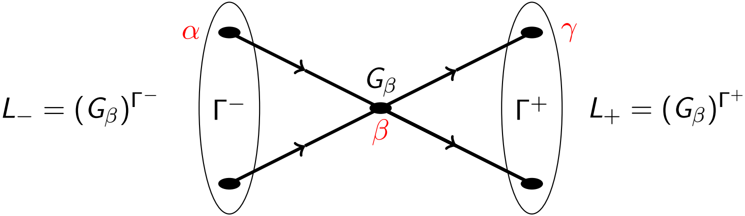
Outline

1. Motivation from digraphs
2. Examples show bounds are best possible
3. Dynkin-Aschbacher classification
4. Proof of the main theorem (Thm 4)
5. Future work



Motivation

- Permutation group $G \leqslant \mathrm{Sym}(\Omega) \rightsquigarrow$ digraph Γ.
- $(\alpha, \beta) \in \Omega \times \Omega$; Arcs of $\Gamma = (\alpha, \beta)^G \rightsquigarrow \Gamma$ is G-arc transitive.
- neighbours $\Gamma^\varepsilon := \Gamma^\varepsilon(\beta)$; local actions $L_\varepsilon := (G_\beta)^{\Gamma^\varepsilon}$



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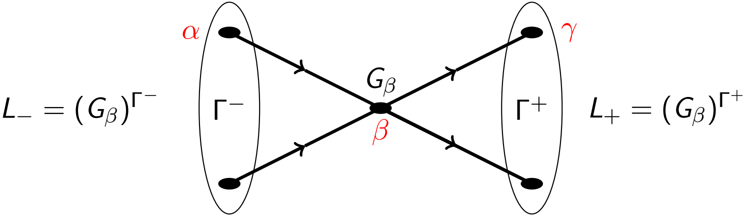

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- Suppose $L_+ = L_-/N$, $N \neq 1$. There are 8^2 possible types for the pair (L_-, L_+) of q.p. groups. Only (HS, AS) and (HC, TW) arise. To eliminate the possibility (HA, HA) it seemed desirable to prove:
- **Theorem [us]** If $G \leqslant \mathrm{GL}(d, p)$ is irreducible, then the number of composition factors of G of order p is at most $d - 1$.

- Definition. If G is a finite group, then let $c_p(G)$ denote the number of composition factors of G that have order p.
- Ex 1. If $G = \mathrm{Sym}(4)$, then $c_2(G) = 3$ and $c_3(G) = 1$.
- Ex 2. $c_p(G) \leqslant \log_p |G|_p$ equality iff G is p-solvable.
- Ex 3. If $G \leqslant \mathrm{GL}(d, p^f)$, then $c_p(G) \leqslant \log_p |\mathrm{GL}(d, p^f)|_p = \binom{d}{2} f$.
- Ex 4. If $G \leqslant \mathrm{Sym}(n)$, then $c_p(G) \leqslant (n-1)/(p-1)$.

- Want. If $G \leqslant \mathrm{GL}(d, p)$ is irreducible, then $c_p(G) \leqslant d - 1$.
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- **Thm 1.** If $G \leqslant \mathrm{GL}(d,q)$ is c.r., then $c_p(G) \leqslant (d-1)f$.
- **Thm 2.** If $G \leqslant \mathrm{GL}(d,q)$ is c.r., then $c_p(G) \leqslant (d-1)f/(p-1)$.
- **Thm 3.** If $G \leqslant \mathrm{GL}(d,q)$ is c.r., then $c_p(G) \leqslant (\frac{3d}{2}-1)/(p-1)$.
- **Thm 4.** If $G \leqslant \mathrm{GL}(d,q)$ is c.r., then $c_p(G) \leqslant (\varepsilon_q d-1)/(p-1)$

$$\text{where } \varepsilon_q = \begin{cases} 4/3 & \text{if } p=2 \text{ and } f \text{ is even (so } q = 4^{f/2}), \\ p/(p-1) & \text{if } q = p \text{ is a Fermat prime,} \\ 1 & \text{otherwise.} \end{cases}$$

Examples show bounds are best possible

- Examples $\rightsquigarrow$ bounds are tight infinitely often.
- Fix $\Gamma_1 \leqslant \mathrm{GL}(k,q)$ and form imprimitive wreath products:
  $\Gamma_2 := \Gamma_1 \wr \mathrm{C}_p \leqslant \mathrm{GL}(kp,q)$, $\Gamma_n = \Gamma_{n-1} \wr \mathrm{C}_p \leqslant \mathrm{GL}(kp^{n-1},q)$ for $n > 1$.

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- Generic $\varepsilon_q = 1$. Let $q = p^f$ and $\Gamma_1 = \mathrm{C}_{p^p-1} \rtimes \mathrm{C}_p \leqslant \mathrm{GL}(p, p)$, so $k = p$. Then $\Gamma_n \leqslant \mathrm{GL}(p^n, p) \leqslant \mathrm{GL}(p^n, q)$ and $c_p(\Gamma_n) = (p^n - 1)/(p - 1) = (d - 1)/(p - 1)$.
- $\varepsilon_q = p/(p - 1)$. If $p = q = 2^m + 1$ is a Fermat prime and Γ_1 is Sylow p-subgroup of $\mathrm{GO}^-(2m, 2)$. If $\Gamma_1 \leqslant \mathrm{GL}(2^m, p) = \mathrm{GL}(p - 1, p)$, then $\Gamma_n \leqslant \mathrm{GL}(d_n, p)$ is irreducible $c_p(\Gamma_n) = (\varepsilon d_n - 1)/(p - 1)$ where $d_n = (p - 1)p^{n-1}$ and $\varepsilon = p/(p - 1) \leqslant 3/2$.
- $\varepsilon_q = 4/3$. Take $p = 2$, $q = 2^2$, and $\Gamma_1 = \mathrm{GU}(3, 2) \leqslant \mathrm{GL}(3, 4)$. Then $\Gamma_n \leqslant \mathrm{GL}(d_n, 4)$ where $d_n = 3 \cdot 2^{n-1}$ is irreducible and $c_2(G) = 2^{n+1} - 1$, so $c_p(G) = (\varepsilon d_n - 1)/(p - 1)$ where $\varepsilon = 4/3$.

Dynkin-Aschbacher classification

Dynkin-Aschbacher Theorem. Every completely reducible subgroup G of $\mathrm{GL}(d,q)$ lies in at least on of the following classes.

- $\mathcal{C}_1$ (reducible subgps) $V = V_1 \oplus V_2$, $G \leqslant \mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$.
- $\mathcal{C}_2$ (imprimitive subgps) $V = V_1 \oplus \cdots \oplus V_r$, $G \leqslant \mathrm{GL}(d/r,q) \wr \mathrm{Sym}(r)$.
- $\mathcal{C}_3$ (ext field subgps) $V = (\mathbb{F}_{q^r})^{d/r}$, and $G \leqslant \mathrm{GL}(d/r,q^r) \rtimes \mathrm{C}_r$.
- $\mathcal{C}_4$ (tensor reducible subgps) $V = V_1 \otimes V_2$ and $G \leqslant \mathrm{GL}(V_1) \otimes \mathrm{GL}(V_2)$.
- $\mathcal{C}_5$ (proper subfield subgps) $G \leqslant \mathrm{GL}(d,q_0) \circ \mathrm{Z}(\mathrm{GL}(d,q))$, $q = q_0^r$.
- $\mathcal{C}_6$ (symplectic type r-groups) $d = r^m$, $R \lhd G \leqslant N_{\mathrm{GL}(d,q)}(R)$ where $R/\mathrm{Z}(R) \cong \mathrm{C}_r^{2m}$ is elementary, and $\Phi(R) \leqslant \mathrm{Z}(R)$.
- $\mathcal{C}_7$ (tensor reducible subgps) $V = V_1 \otimes \cdots \otimes V_r$ and $G \leqslant \mathrm{GL}(d^{1/r},q) \wr \mathrm{Sym}(r)$.
- $\mathcal{C}_8$ (classical groups) preserves symplectic, unitary, or orthogonal form and contains $\mathrm{Sp}(V)'$, $\mathrm{SU}(V)$, or $\Omega^{\varepsilon}(V)$ resp., where $\varepsilon \in \{\pm, \circ\}$.
- $\mathcal{C}_9$ (nearly simple) $Z := \mathrm{Z}(G)$, $\mathrm{socle}(G/Z) = N/Z$ is almost simple and absolutely irreducible.

Proof of the main theorem (Thm 4)

- Induction on (d, q) ordered lexicographically

$$(d_1, q_1) < (d_2, q_2) \quad \text{if } d_1 < d_2 \text{ or } d_1 = d_2 \text{ and } q_1 < q_2.$$

- Simple cases:
- $\mathcal{C}_1$. Then $G \leqslant \mathrm{GL}(d_1, q) \times \mathrm{GL}(d_2, q)$, so $G \leqslant G_1 \times G_2$ and

$$c_p(G) \leqslant c_p(G_1) + c_p(G_2) \leqslant \frac{\varepsilon_q d_1 - 1}{p-1} + \frac{\varepsilon_q d_2 - 1}{p-1} = \frac{\varepsilon_q(d_1 + d_2) - 2}{p-1} < \frac{\varepsilon_q d - 1}{p-1}.$$

- $\mathcal{C}_2$. Then $V = V_1 \oplus \cdots \oplus V_r$, and $G \leqslant \mathrm{GL}(d/r, q) \wr \mathrm{Sym}(r)$, so $G \leqslant G_1 \wr G_2$ and

$$c_p(G) \leqslant rc_p(G_1) + c_p(G_2) \leqslant \frac{r(\varepsilon_q d/r - 1)}{p-1} + \frac{r-1}{p-1} = \frac{\varepsilon_q d - 1}{p-1}.$$

Proof of the main theorem (Thm 4)

- $\mathcal{C}_3$. compare ε_q and ε_{q^r}.
- $\mathcal{C}_4$. Like $\mathcal{C}_1$; $\mathcal{C}_7$ like $\mathcal{C}_2$; $\mathcal{C}_5$ induction.
- $\mathcal{C}_8$. Easy.
- $\mathcal{C}_6$. Harder case. Number theory $|G|$ small $\rightsquigarrow c_p(G)$ small.
- $\mathcal{C}_9$. Hardest case. $T = N/Z$ simple, $|G/N|$ divides $|\mathrm{Out}(T)|$, $c_p(G) = \log_p |G/N|_p \leqslant \log_p |\mathrm{Out}(T)|_p$. Most difficulties when $T = L(q')$ simple of Lie-type and $q' = p^{f'}$.
  Example: Suppose $N = \mathrm{SL}(k,q) \leqslant \mathrm{GL}(d,q)$ where $d = \binom{k}{2}$. Why doesn't the normalizer G of N in $\mathrm{GL}(d,q)$ include many field automorphisms? Recall $q = p^f$. What if $\log_p(f_p) > (\varepsilon_q d - 1)/(p-1)$?

Future work

- Are the given examples the only examples matching the bounds?
- Are there smaller bounds for $c_p(G)$ for completely reducible subgroups of symplectic, unitary, orthogonal groups?
- What about bounds for $c_T(G)$ when T is a simple group? (Small progress.)
- What if the prime $p \neq \mathrm{char}(\mathbb{F}_q)$? If $G \leqslant \mathrm{GL}(d, q)$ is completely reducible, then find *sharp* upper bounds for $c_p(G)$. (Partially solved.)

Application. Limit the local symmetries of digraphs, and construct new highly symmetric examples.

Research Associate/Fellow position at UWA

For details see Cheryl Praeger, me, or

http://external.jobs.uwa.edu.au/cw/en/job/499094/

The CMSC and Western Australia are remarkable places!