Primitive actions of groups of intermediate word growth

Alejandra Garrido
joint work with Dominik Francoeur (University of Geneva)

University of Düsseldorf

Groups St Andrews 2017 in Birmingham
11 August 2017

Primitive groups and Maximal subgroups

Primitive permutation actions = "atoms" of permutation actions
$G \curvearrowright X$ (transitive) is primitive $\Leftrightarrow$ point stabilizers are maximal.

General question

Given a group (not as permutation group), what are its primitive permutation representations? i.e. What are its maximal subgroups?

If G is finitely generated, every proper subgroup is contained in a maximal one.

First basic questions

Does a given finitely generated group contain maximal subgroups of infinite index? i.e. Can the group act primitively on an infinite set? Is this action faithful?

Some known results

Let $\mathcal{IP}$ denote the class of f.g. groups with some maximal subgroup of infinite index.

$\notin \mathcal{IP}$

- nilpotent groups (normaliser condition)
- virtually soluble linear groups [Margulis+Soifer, '81]

$\in \mathcal{IP}$

- free groups [McDonough, '77]
- not v.s. linear groups [Margulis+Soifer, '81]
- mapping class groups, hyperbolic groups, other "geometric" groups (with appropriate caveats) [Gelander+Glasner, '07]

Big and small groups: word growth

Definition

The growth function $\gamma_G(n)$ of G w.r.t finite generating set S gives the number of elements of G of S-length $\leq n$.

Up to equivalence relation, $\gamma_G(n)$ does not depend on S.

Big and small groups: word growth

Definition

The growth function $\gamma_G(n)$ of G w.r.t finite generating set S gives the number of elements of G of S-length $\leq n$.

Up to equivalence relation, $\gamma_G(n)$ does not depend on S.

Types of growth (up to equivalence):

- $\gamma_G(n) \approx n^a, a \in \mathbb{N} \Leftrightarrow$ virtually nilpotent [Wolf,Bass,Guivarch; Gromov]
- $\gamma_G(n) \approx \exp(n)$ e.g. free groups, not v.s. linear groups [Tits alternative, '72]
- $\gamma_G(n)$ is super-polynomial and sub-exponential: intermediate growth [first examples by Grigorchuk, '85]

Big and small groups: word growth

Definition

The growth function $\gamma_G(n)$ of G w.r.t finite generating set S gives the number of elements of G of S-length $\leq n$.

Up to equivalence relation, $\gamma_G(n)$ does not depend on S.

Types of growth (up to equivalence):

- $\gamma_G(n) \approx n^a, a \in \mathbb{N} \Leftrightarrow$ virtually nilpotent [Wolf,Bass,Guivarch; Gromov]
- $\gamma_G(n) \approx \exp(n)$ e.g. free groups, not v.s. linear groups [Tits alternative, '72]
- $\gamma_G(n)$ is super-polynomial and sub-exponential: intermediate growth [first examples by Grigorchuk, '85]

All known examples of groups $\in \mathcal{IP}$ are of exponential growth.

Big and small groups: word growth

Definition

The growth function $\gamma_G(n)$ of G w.r.t finite generating set S gives the number of elements of G of S-length $\leq n$.

Up to equivalence relation, $\gamma_G(n)$ does not depend on S.

Types of growth (up to equivalence):

- $\gamma_G(n) \approx n^a, a \in \mathbb{N} \Leftrightarrow$ virtually nilpotent [Wolf,Bass,Guivarch; Gromov]
- $\gamma_G(n) \approx \exp(n)$ e.g. free groups, not v.s. linear groups [Tits alternative, '72]
- $\gamma_G(n)$ is super-polynomial and sub-exponential: intermediate growth [first examples by Grigorchuk, '85]

All known examples of groups $\in \mathcal{IP}$ are of exponential growth.

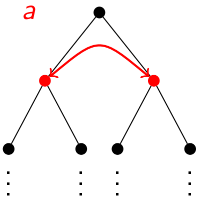
Question (Cornulier, '06)

Are there groups of intermediate growth in $\mathcal{IP}$?

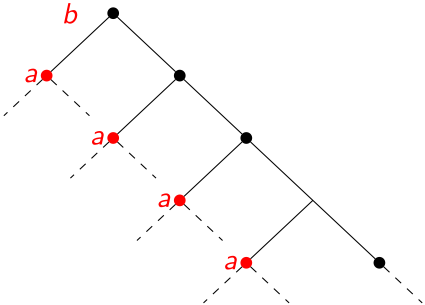
Action of D_∞ on binary rooted tree

Let T =rooted, infinite binary tree, Aut T=its group of automorphisms. Consider $D_\infty = \langle a, b\rangle \leq \text{Aut } T$:

a ="swap" on level 1



$b = (a, b)$



Fragmentations of D_∞

Two subgroups of Aut T

$$\begin{aligned} G_1 &= \langle a, \beta, \gamma, \delta \rangle & G_2 &= \langle a, b, c, d \rangle \\ \beta &= (a, \gamma) & b &= (a, b) \\ \gamma &= (a, \delta) & c &= (a, d) \\ \delta &= (1, \beta) & d &= (1, c) \end{aligned}$$

Fragmentations of D_∞

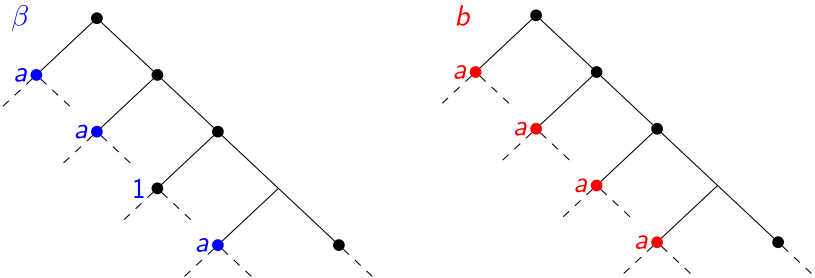
Two subgroups of Aut T

$$G_1 = \langle a, \beta, \gamma, \delta \rangle \qquad G_2 = \langle a, b, c, d \rangle$$

$$\beta = (a, \gamma) \qquad b = (a, b)$$

$$\gamma = (a, \delta) \qquad c = (a, d)$$

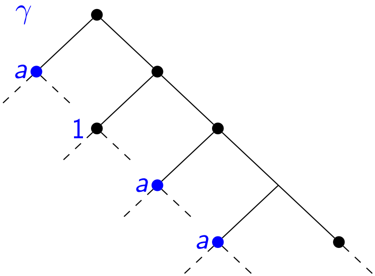
$$\delta = (1, \beta) \qquad d = (1, c)$$

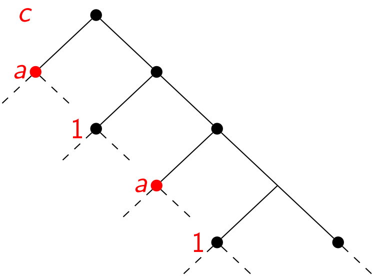


Fragmentations of D_∞

Two subgroups of Aut T

$$G_1 = \langle a, \beta, \gamma, \delta \rangle \qquad G_2 = \langle a, b, c, d \rangle$$
$$\beta = (a, \gamma) \qquad b = (a, b)$$
$$\gamma = (a, \delta) \qquad c = (a, d)$$
$$\delta = (1, \beta) \qquad d = (1, c)$$





Fragmentations of D_∞

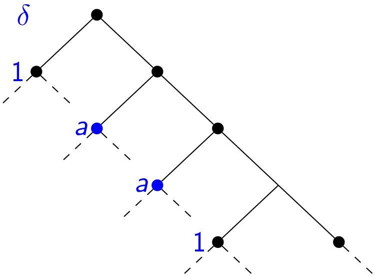
Two subgroups of Aut T

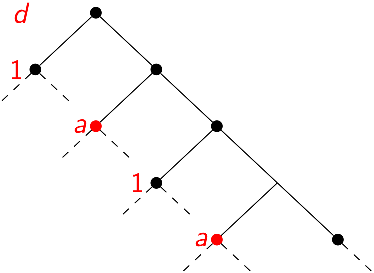
$$G_1 = \langle a, \beta, \gamma, \delta \rangle \qquad G_2 = \langle a, b, c, d \rangle$$

$$\beta = (a, \gamma) \qquad b = (a, b)$$

$$\gamma = (a, \delta) \qquad c = (a, d)$$

$$\delta = (1, \beta) \qquad d = (1, c)$$





Properties of these two examples

$G_1 = \langle a, \beta, \gamma, \delta \rangle$

- "Grigorchuk group"

$G_2 = \langle a, b, c, d \rangle$

- "Grigorchuk–Erschler group"

Properties of these two examples

$G_1 = \langle a, \beta, \gamma, \delta \rangle$

- "Grigorchuk group"
- intermediate growth [Grigorchuk, '85]

$G_2 = \langle a, b, c, d \rangle$

- "Grigorchuk–Erschler group"
- intermediate growth [Grigorchuk, '85]

Properties of these two examples

$G_1 = \langle a, \beta, \gamma, \delta \rangle$

- "Grigorchuk group"
- intermediate growth [Grigorchuk, '85]
- torsion, 2-group

$G_2 = \langle a, b, c, d \rangle$

- "Grigorchuk–Erschler group"
- intermediate growth [Grigorchuk, '85]
- not torsion, $\langle a, b \rangle \cong D_\infty$

Properties of these two examples

$G_1 = \langle a, \beta, \gamma, \delta \rangle$

- "Grigorchuk group"
- intermediate growth [Grigorchuk, '85]
- torsion, 2-group
- $\notin \mathcal{IP}$ [Pervova , '00]

$G_2 = \langle a, b, c, d \rangle$

- "Grigorchuk–Erschler group"
- intermediate growth [Grigorchuk, '85]
- not torsion, $\langle a, b \rangle \cong D_\infty$

Properties of these two examples

$G_1 = \langle a, \beta, \gamma, \delta \rangle$

- "Grigorchuk group"
- intermediate growth [Grigorchuk, '85]
- torsion, 2-group
- $\notin \mathcal{IP}$ [Pervova , '00]

$G_2 = \langle a, b, c, d \rangle$

- "Grigorchuk–Erschler group"
- intermediate growth [Grigorchuk, '85]
- not torsion, $\langle a, b \rangle \cong D_\infty$
- $\in \mathcal{IP}$ [F+G, '16]

Properties of these two examples

$G_1 = \langle a, \beta, \gamma, \delta \rangle$

- "Grigorchuk group"
- intermediate growth [Grigorchuk, '85]
- torsion, 2-group
- $\notin \mathcal{IP}$ [Pervova , '00]

$G_2 = \langle a, b, c, d \rangle$

- "Grigorchuk–Erschler group"
- intermediate growth [Grigorchuk, '85]
- not torsion, $\langle a, b \rangle \cong D_\infty$
- $\in \mathcal{IP}$ [F+G, '16]

Actually, we prove this for a larger family of "siblings of Grigorchuk's group" defined by Šunić. They are all self-similar fragmentations of $D_\infty \leq \mathrm{Aut}\, T$ and are of intermediate growth.
We show that the non-torsion ones (=those containing D_∞) are in $\mathcal{IP}$, by finding their maximal subgroups.

Main results [Francoeur + G, '16]

Theorem 1

Let $G_2 = \langle a, b, c, d \rangle$ be as above (or one of its non-torsion "siblings"). Its maximal subgroups are:

Main results [Francoeur + G, '16]

Theorem 1

Let $G_2 = \langle a, b, c, d \rangle$ be as above (or one of its non-torsion "siblings"). Its maximal subgroups are:

- of index 2 (7 of them);

Main results [Francoeur + G, '16]

Theorem 1

Let $G_2 = \langle a, b, c, d \rangle$ be as above (or one of its non-torsion "siblings"). Its maximal subgroups are:

- of index 2 (7 of them);
- $H(q) = \langle (ab)^q, b, c, d \rangle$ for q odd prime, of infinite index ($\aleph_0$ of them);

N.B. $\langle (ab)^q, b \rangle$ is a maximal subgroup of D_∞ for each odd prime q.

Main results [Francoeur + G, '16]

Theorem 1

Let $G_2 = \langle a, b, c, d \rangle$ be as above (or one of its non-torsion "siblings"). Its maximal subgroups are:

- of index 2 (7 of them);
- $H(q) = \langle (ab)^q, b, c, d \rangle$ for q odd prime, of infinite index ($\aleph_0$ of them);
- ('17+) conjugates of $H(q)$ in G.

N.B. $\langle (ab)^q, b \rangle$ is a maximal subgroup of D_∞ for each odd prime q.

Main results [Francoeur + G, '16]

Theorem 1

Let $G_2 = \langle a, b, c, d \rangle$ be as above (or one of its non-torsion "siblings"). Its maximal subgroups are:

- of index 2 (7 of them);
- $H(q) = \langle (ab)^q, b, c, d \rangle$ for q odd prime, of infinite index ($\aleph_0$ of them);
- ('17+) conjugates of $H(q)$ in G.

N.B. $\langle (ab)^q, b \rangle$ is a maximal subgroup of D_∞ for each odd prime q.
Additional fact: Each $H(q)$ is conjugate to G_2 in Aut T.

Main results [Francoeur + G, '16]

Theorem 1

Let $G_2 = \langle a, b, c, d \rangle$ be as above (or one of its non-torsion "siblings").
It has $\aleph_0$ maximal subgroups of infinite index. (explicit:
$H(q) = \langle (ab)^q, b, c, d \rangle$)

Main results [Francoeur + G, '16]

Theorem 1

Let $G_2 = \langle a, b, c, d \rangle$ be as above (or one of its non-torsion "siblings"). It has $\aleph_0$ maximal subgroups of infinite index. (explicit: $H(q) = \langle (ab)^q, b, c, d \rangle$)

Šunić also defined odd-prime siblings of Grigorchuk's group, they act on the p-regular tree, where p is an odd prime, and have a similar definition to the one we saw.

Main results [Francoeur + G, '16]

Theorem 1

Let $G_2 = \langle a, b, c, d\rangle$ be as above (or one of its non-torsion "siblings"). It has $\aleph_0$ maximal subgroups of infinite index. (explicit: $H(q) = \langle (ab)^q, b, c, d\rangle$)

Šunić also defined odd-prime siblings of Grigorchuk's group, they act on the p-regular tree, where p is an odd prime, and have a similar definition to the one we saw.

Theorem 2

All groups in Šunić's family (for all primes) are just infinite (all proper quotients are finite).

Main results [Francoeur + G, '16]

Theorem 1

Let $G_2 = \langle a, b, c, d \rangle$ be as above (or one of its non-torsion "siblings"). It has $\aleph_0$ maximal subgroups of infinite index. (explicit: $H(q) = \langle (ab)^q, b, c, d \rangle$)

Šunić also defined odd-prime siblings of Grigorchuk's group, they act on the p-regular tree, where p is an odd prime, and have a similar definition to the one we saw.

Theorem 2

All groups in Šunić's family (for all primes) are just infinite (all proper quotients are finite).

Corollary

G_2 is a primitive permutation group and has trivial Frattini subgroup (Cfr. G_1 has Frattini subgroup of finite index).

When does a maximal subgroup have infinite index?

Definition

The profinite topology of a group G has $\{N \lhd G \mid |G : N| < \infty\}$ as base of neighbourhoods of the identity.

$H \leq G$ is dense if $HN = G$ for every $N \lhd G$ with $|G : N| < \infty$.

When does a maximal subgroup have infinite index?

Definition

The profinite topology of a group G has $\{N \lhd G \mid |G : N| < \infty\}$ as base of neighbourhoods of the identity.

$H \leq G$ is dense if $HN = G$ for every $N \lhd G$ with $|G : N| < \infty$.

Fact

A maximal subgroup is of infinite index if and only if it is dense in the profinite topology.

Profinite vs Aut T topology

Step 1: Show that $H(q) \leq G_2$ is dense in the profinite topology.

Profinite vs Aut T topology

Step 1: Show that $H(q) \leq G_2$ is dense in the profinite topology.
Problem: This means knowing ALL finite index subgroups. How can we get a hold of them?

Profinite vs Aut T topology

Step 1: Show that $H(q) \leq G_2$ is dense in the profinite topology.
Problem: This means knowing ALL finite index subgroups. How can we get a hold of them? Obvious candidates: level stabilizers $\{\mathrm{St}_{G_2}(n) \mid n \in \mathbb{N}\}$.

Profinite vs Aut T topology

Step 1: Show that $H(q) \leq G_2$ is dense in the profinite topology.
Problem: This means knowing ALL finite index subgroups. How can we get a hold of them? Obvious candidates: level stabilizers $\{\mathrm{St}_{G_2}(n) \mid n \in \mathbb{N}\}$.

Definition

A group $G \leq \mathrm{Aut}\, T$ has the congruence subgroup property if every finite index subgroup of G contains some level stabilizer $\mathrm{St}_G(n)$.

Profinite vs Aut T topology

Step 1: Show that $H(q) \leq G_2$ is dense in the profinite topology.
Problem: This means knowing ALL finite index subgroups. How can we get a hold of them? Obvious candidates: level stabilizers $\{\mathrm{St}_{G_2}(n) \mid n \in \mathbb{N}\}$.

Definition

A group $G \leq \mathrm{Aut}\, T$ has the congruence subgroup property if every finite index subgroup of G contains some level stabilizer $\mathrm{St}_G(n)$.

Theorem (Francoeur+G, '16)

All Šunić groups (and G_2 in particular) have the congruence subgroup property.
In fact, every normal subgroup contains a level stabilizer.

Dense subgroups in Aut T

Step 1a: $H(q) \leq G_2$ satisfies $H(q)\,\mathrm{St}_{G_2}(n) = G_2$ for each $n \in \mathbb{N}$.

Dense subgroups in Aut T

Step 1a: $H(q) \leq G_2$ satisfies $H(q)\,\mathrm{St}_{G_2}(n) = G_2$ for each $n \in \mathbb{N}$.

Since q is odd (coprime with 2), taking a power of $(ab)^q$ produces the same action as ab on the nth level of the tree (Euclidean algorithm).

Dense subgroups in Aut T

Step 1a: $H(q) \leq G_2$ satisfies $H(q)\,\mathrm{St}_{G_2}(n) = G_2$ for each $n \in \mathbb{N}$.

Since q is odd (coprime with 2), taking a power of $(ab)^q$ produces the same action as ab on the nth level of the tree (Euclidean algorithm).

Corollary

Let q be an odd integer, then $H(q) = \langle (ab)^q, b, c, d \rangle$ is a dense subgroup of G_2 for the profinite topology.

$H(q)$ is proper

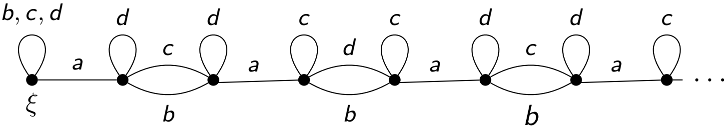
$H(q)$ is proper

Step 2: Show that $H(q)$ is a proper subgroup (indeed, taking the same definition in G_1 yields $H(q) = G_1$).

$H(q)$ is proper

Step 2: Show that $H(q)$ is a proper subgroup (indeed, taking the same definition in G_1 yields $H(q) = G_1$).

Look at actions of $H(q)$ and G_2 on boundary of tree T. Suffices to consider orbit of ξ =rightmost ray. Thanks to copy of dihedral group $\langle a, b \rangle$, the orbit of ξ under G_2 is isomorphic to $\mathbb{Z}$. But the orbit under $H(q)$ is strictly smaller (corresponds to $q\mathbb{Z}$):



Maximal subgroups are conjugate to $H(q)$

Steps 3+4: Some technical work, using techniques similar to those of Pervova to show

Theorem

Let q be an odd prime, then $H(q)$ is maximal and of infinite index in G_2.

Maximal subgroups are conjugate to $H(q)$

Steps 3+4: Some technical work, using techniques similar to those of Pervova to show

Theorem

Let q be an odd prime, then $H(q)$ is maximal and of infinite index in G_2.

Theorem

All maximal subgroups of infinite index in G_2 are conjugates of some $H(q)$.

Maximal subgroups are conjugate to $H(q)$

Steps 3+4: Some technical work, using techniques similar to those of Pervova to show

Theorem

Let q be an odd prime, then $H(q)$ is maximal and of infinite index in G_2.

Theorem

All maximal subgroups of infinite index in G_2 are conjugates of some $H(q)$.

Questions

- Is G_2 oligomorphic? It's not of finite sub-degree [follows from Wesolek, '16]
- A more conceptual proof of maximality and 'uniqueness 'of $H(q)$?

Thank you!