Cohomology of finite p-groups and coclass theory

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Answer: I do not know.
Fact: If the group property is good, cohomology should not be able to tell them apart.
That is, given an infinite family of p-groups $\{G_i\}_{i\geq 0}$ with a 'good' common property, deduce that there are finitely many isomorphisms types of algebras in $\{H^*(G_i;\mathbb{F}_p)\}$.

Abelian p-groups

Let $K \cong C_{p^{i_1}} \times \cdots \times C_{p^{i_d}}$ be an abelian p-group. Then,

$$H^*(K; \mathbb{F}_p) \cong \begin{cases} \mathbb{F}_2[y_1, \ldots, y_d] & \text{if } p = 2, i_l = 1, \\ \Lambda(y_1, \ldots, y_d) \otimes \mathbb{F}_p[x_1, \ldots, x_d] & \text{otherwise,} \end{cases}$$

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$$H^*(D_{2^n};\mathbb{F}_2)_{n\geq 3} \cong \mathbb{F}_2[a,b,x]/(ab) \text{ with } |a|=|b|=1, |x|=2,$$
$$H^*(Q_{2^n};\mathbb{F}_2)_{n\geq 4} \cong \mathbb{F}_2[a,b,y]/(a^2+ab, y^3) \text{ with } |a|=|b|=1, |y|=4,$$
$$H^*(SD_{2^n};\mathbb{F}_2)_{n\geq 5} \cong \mathbb{F}_2[a,b,z,y]/(a^2+ab, a^3, az, z^2(a^2+b^2)y),$$
$$\text{where } |a|=|b|=1, |z|=3, |y|=4.$$

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Let p be an odd prime, let k be a field of characteristic p and let c be an integer. Then, there are only finitely many isomorphism types of cohomology algebras in the collection $H^*(G; k)$ when G runs over the p-groups of coclass c.

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Let p be a prime number and let c be an integer. Then, there are only finitely many isomorphism types of cohomology algebras $H^*(G;\mathbb{F}_p)$ when G runs over the non-twisted p-groups of coclass c.

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For instance, all 2-groups of fixed coclass are non-twisted.

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3. Some counting arguments for cohomology algebras using spectral sequences.

Our result is based on:

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Our result is based on:

1. A *reformulated* classification of p-groups by their coclass (The Structure Theorem of Leedham-Green),
2. The fact that there finitely many cohomology algebras for all 2-groups of maximal nilpotency class (explicit isomorphisms),
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Our result is based on:

1. A *reformulated* classification of p-groups by their coclass (The Structure Theorem of Leedham-Green),
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Our result is based on:

1. A *reformulated* classification of p-groups by their coclass (The Structure Theorem of Leedham-Green),
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3. *Refined* counting arguments for cohomology algebras using spectral sequences.

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Theorem (The Structure Theorem, Leedham-Green, 1994)

Let p be a prime number, let c be an integer and let G be a p-group of coclass c. Then, there exist a normal subgroup N of G and a function $f(p, c)$ such that $|N| \leq f(p, c)$ and G/N is constructible.

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So, for all $n \geq 4$,

$$D_{2^{n-1}} \cong Q_{2^n}/C_2 \text{ and } D_{2^{n-1}} \cong SD_{2^n}/C_2.$$

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- *R has coclass at least x.*

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1. Introduction
2. The Structure Theorem and Constructible groups
   - Uniserial p-adic space groups
   - Constructible groups
3. Final remarks

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2. Twisted: quotients + 'twist' on the group operation of K (technical).
3. We say that a p-group of fixed coclass is *non-twisted* if for some normal subgroup $N \leq G$ of bounded order, G/N is constructible non-twisted. Otherwise, we say that G is *twisted*.

Picture: non-twisted case

Let G be a p-group of coclass c.

$$\begin{array}{ccccc} & & & & K \rtimes P \\ & & & & \uparrow \\ N & \longrightarrow & G & \longrightarrow & G/N \cong R/U \end{array}$$

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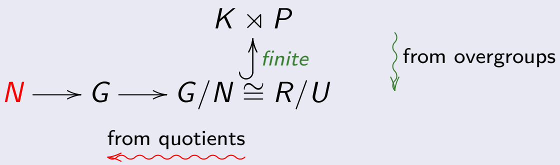

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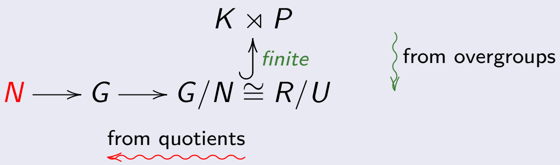

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- Current work: delete the condition on the nilpotency class (being smaller than p).

THANK YOU FOR YOUR ATTENTION!