On splitting of the normalizer of a maximal torus in groups of Lie type

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Example 1

Let $\overline{G} = \mathrm{SL}_2(\overline{\mathbb{F}}_p)$ be the special linear group of degree 2 over $\overline{\mathbb{F}}_p$. Then $\overline{T} = \{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \lambda \in \overline{\mathbb{F}}_p^*\}$ is a maximal torus of $\overline{G}$. The normalizer $N_{\overline{G}}(\overline{T})$ is the group of all monomial matrices of $\overline{G}$ and $N_{\overline{G}}(\overline{T})/\overline{T} \simeq \mathrm{Sym}_2$. But $\overline{G}$ contains only one element of order two: $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, and this element lies in $\overline{T}$. Hence, $N_{\overline{G}}(\overline{T})$ does not split over $\overline{T}$.

Example 2

Let $\overline{G} = \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ be the general linear group of degree n over $\overline{\mathbb{F}}_p$. Then $\overline{T} = D_n(\overline{\mathbb{F}}_p)$ is a maximal torus of $\overline{G}$. The normalizer $N_{\overline{G}}(\overline{T})$ is the group of all monomial matrices of $\overline{G}$ and $N_{\overline{G}}(\overline{T})/\overline{T} \simeq \mathrm{Sym}_n$.
There is a canonical embedding of Sym_n into the group of all monomial matrices of $\overline{G}$. If $\overline{H}$ is an image of Sym_n under this embedding, then $\overline{H}$ is a complement for $\overline{T}$ in $N_{\overline{G}}(\overline{T})$.

Since the center $Z(\overline{G})$ of $\overline{G}$ is contained in $\overline{T}$, then a maximal torus of $\mathrm{PGL}_n(\overline{\mathbb{F}}_p)$ also has a complement in their normalizer. Moreover, $\mathrm{PGL}_n(\overline{\mathbb{F}}_p) \simeq \mathrm{PSL}_n(\overline{\mathbb{F}}_p)$ and the same is true for $\mathrm{PSL}_n(\overline{\mathbb{F}}_p)$.

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Problems

Let $\overline{G}$ be a simple connected linear algebraic group over the algebraic closure $\overline{\mathbb{F}}_p$ of a finite field of positive characteristic p. Let σ be a Steinberg endomorphism and $\overline{T}$ a maximal σ-invariant torus of $\overline{G}$. It's well known that all the maximal tori are conjugated in $\overline{G}$ and the quotient $N_{\overline{G}}(\overline{T})/\overline{T}$ is isomorphic to the Weyl group W of $\overline{G}$. The following problem arises.

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Describe the groups $\overline{G}$ in which $N_{\overline{G}}(\overline{T})$ splits over $\overline{T}$.

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A similar problem arises in finite groups G of Lie type. Let $T = \overline{T} \cap G$ be a maximal torus in a finite group of Lie type G, $N(G,T) = N_{\overline{G}}(\overline{T}) \cap G$ an algebraic normalizer of G. Notice that $N(G,T) \leqslant N_G(T)$, but the equality is not true in general.

Problem 2

Describe the groups G and their maximal tori T in which $N(G,T)$ splits over T.

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History

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An answer to Problem 1 for simple Lie groups was given in M. Curtis, A. Wiederhold, B. Williams, "Normalizers of maximal tori" // Springer, Berlin, 1974, Lecture Notes in Math., V. 418, 31–47.

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The answer for Problem 1 is in the following table:

Group	Conditions of existence of a complement
$\mathrm{SL}_n(\overline{\mathbb{F}}_p)$	$p = 2$ or n is odd
$\mathrm{PSL}_n(\overline{\mathbb{F}}_p)$	No conditions
$\mathrm{Sp}_{2n}(\overline{\mathbb{F}}_p)$	$p = 2$
$\mathrm{PSp}_{2n}(\overline{\mathbb{F}}_p)$	$p = 2$ or $n \leqslant 2$
$\mathrm{SO}_{2n+1}(\overline{\mathbb{F}}_p)$	No conditions
$\mathrm{SO}_{2n}(\overline{\mathbb{F}}_p)$	No conditions
$\mathrm{PSO}_{2n}(\overline{\mathbb{F}}_p)$	No conditions
$G_2(\overline{\mathbb{F}}_p)$	No conditions
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Let $O^{p'}(\overline{G}_\sigma) \leqslant G \leqslant \overline{G}_\sigma$ be a finite group of Lie type. Two maximal tori in G are not necessary conjugate in G. Let W be a Weyl group of $\overline{G}$, π a natural homomorphism from $\overline{N} = N_{\overline{G}}(\overline{T})$ into W. Two elements w_1, w_2 are called σ-conjugate if $w_1 = (w^{-1})^\sigma w_2 w$ for some element w of W.

Proposition

There is a bijection between the G-classes of σ-stable maximal tori of $\overline{G}$ and the σ-conjugacy classes of W.

Define $C_{W,\sigma}(w) = \{x \in W | (x^{-1})^\sigma wx = w\}$.

Proposition

Let $g^\sigma g^{-1} \in \overline{N}$ and $\pi(g^\sigma g^{-1}) = w$. Then

$$(N_{\overline{G}}(\overline{T}^g))_\sigma / (\overline{T}^g)_\sigma \simeq C_{W,\sigma}(w).$$

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In case of linear group $W \simeq \mathrm{Sym}_n$ and the σ-conjugacy classes $C_{W,\sigma}(w)$ of W are coincide with ordinary conjugacy classes of symmetric group. Each such class corresponds to the cycle-type $(n_1)(n_2)\ldots(n_m)$. Let $\{n_1, \ldots, n_m\}$ be a partition of n. We assume that

$$n_1 = \ldots = n_{l_1} < \ldots < n_{l_1+\ldots+l_{r-1}+1} = \ldots = n_{l_1+\ldots+l_r}$$

and $a_1 = n_{l_1} l_1, a_2 = n_{l_1+l_2} l_2, \ldots, a_r = n_{l_1+\ldots+l_r} l_r$.

Theorem

Let T be a maximal torus of $G = \mathrm{SL}_n(q)$ with the cycle-type $(n_1)(n_2)\ldots(n_m)$. Then T has a complement in N if and only if q is even or a_i is odd for some $1 \leqslant i \leqslant r$.

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Symplectic and orthogonal groups

Let $n = n' + n''$, $\{n_1, \dots, n_k\}$ and $\{n_{k+1}, \dots, n_m\}$ be partitions of n' and n'', respectively. A set $\{-n_1, \dots, -n_k, n_{k+1}, \dots, n_m\}$ will be called a cycle-type and denoted by $(\overline{n_1}) \dots (\overline{n_k})(n_{k+1}) \dots (n_m)$. As above we assume that

$$n_1 = \ldots = n_{l_1} < \ldots < n_{l_1+\ldots+l_{r-1}+1} = \ldots = n_{l_1+\ldots+l_r}$$

Let $a_1 = n_{l_1} l_1, a_2 = n_{l_1+l_2} l_2, \dots, a_r = n_{l_1+\ldots+l_r} l_r$.

Theorem

Let q be a power of a prime p. Let $\overline{T}$ a maximal σ-invariant torus of $\overline{G}$, T a corresponding maximal torus of G with the cycle-type $(\overline{n_1})\dots(\overline{n_k})(n_{k+1})\dots(n_m)$ and $m > 4$. Then

Group	Conditions of existence of a complement
$PSp_{2n}(q)$	q is even
$\Omega_{2n+1}(q)$	$q \equiv 1 \pmod 4$
	a_i is odd for some $1 \leqslant i \leqslant r$
	$q \equiv 3 \pmod 4$ and n_i is even for all $1 \leqslant i \leqslant m$
$P\Omega^+_{2n}(q)$	$q \equiv 1 \pmod 4$
	a_i is odd for some $1 \leqslant i \leqslant r$
	$q \equiv 3 \pmod 4$ and n_i is even for all $1 \leqslant i \leqslant m$
$PSL_n(q)$	q is even
	a_i is odd for some $1 \leqslant i \leqslant r$
	$(n)_2 < (q-1)_2$

The answer for Problem 2 for groups $E_6(q)$ is in the following table:

No	Representative ω	$\lvert\omega\rvert$	$\lvert C_W(\omega)\rvert$	Structure of $C_W(w)$	Torus T	$\rtimes$
1	1	1	51840	$O_5(3):Z_2$	$(q-1)^6$	—
2	ω_1	2	1440	$S_2 \times S_6$	$(q-1)^4 \times (q^2-1)$	—
3	$\omega_1\omega_2$	2	192	$D_8 \times S_4$	$(q-1)^2 \times (q^2-1)^2$	—
4	$\omega_3\omega_1$	3	216	$Z_3 \times (S_3^2 : Z_2)$	$(q-1)^3 \times (q^3-1)$	+
5	$\omega_2\omega_3\omega_5$	2	96	$Z_2 \times Z_2 \times S_4$	$(q^2-1)^3$	—
6	$\omega_1\omega_3\omega_5$	6	36	$Z_6 \times S_3$	$(q-1) \times (q^2-1) \times (q^3-1)$	+
7	$\omega_1\omega_3\omega_4$	4	32	$Z_4 \times D_8$	$(q-1)^2 \times (q^4-1)$	—
8	$\omega_1\omega_4\omega_6\omega_{36}$	2	1152		$(q+1)^2 \times (q^2-1)^2$	—
9	$\omega_1\omega_2\omega_3\omega_5$	6	24	$Z_3 \times D_8$	$(q^2-1) \times (q+1)(q^3-1)$	+
10	$\omega_1\omega_5\omega_3\omega_6$	3	108	$Z_3 \times S_3 \times S_3$	$(q-1) \times (q^2+q+1) \times (q^3-1)$	+
11	$\omega_1\omega_4\omega_6\omega_3$	4	16	$Z_4 \times Z_2 \times Z_2$	$(q^2-1) \times (q^4-1)$	—
12	$\omega_1\omega_4\omega_3\omega_2$	5	10	$Z_2 \times Z_5$	$(q-1) \times (q^5-1)$	+
13	$\omega_3\omega_2\omega_5\omega_4$	6	36	$Z_6 \times S_3$	$(q^2-1) \times (q-1)(q^3+1)$	+
14	$\omega_3\omega_2\omega_4\omega_{14}$	4	96	$SL_2(3):Z_4$	$(q-1)(q^2+1)^2$	—
15	$\omega_1\omega_5\omega_3\omega_6\omega_2$	6	36	$Z_6 \times S_3$	$(q^2+q+1) \times (q+1)(q^3-1)$	+
16	$\omega_1\omega_4\omega_6\omega_3\omega_{36}$	4	96	$Z_4 \times S_4$	$(q+1)^2 \times (q^4-1)$	—
17	$\omega_1\omega_4\omega_5\omega_3\omega_{36}$	10	10	Z_{10}	$(q+1)(q^5-1)$	+
18	$\omega_1\omega_4\omega_6\omega_3\omega_5$	6	12	$Z_6 \times Z_2$	$(q^2+q+1) \times (q-1)(q^3+1)$	+
19	$\omega_2\omega_5\omega_3\omega_4\omega_6$	8	8	Z_8	$(q^2-1)(q^4+1)$	+
20	$\omega_{20}\omega_5\omega_4\omega_3\omega_2$	12	12	Z_{12}	$(q-1)(q^2+1)(q^3+1)$	+
21	$\omega_1\omega_5\omega_2\omega_3\omega_6\omega_{36}$	3	648	$(((Z_3^2):Z_3):Q_8):Z_3$	$(q^2+q+1)^3$	+
22	$\omega_1\omega_4\omega_6\omega_3\omega_5\omega_{36}$	6	36	$Z_6 \times S_3$	$(q+1) \times (q^5+q^4+q^3+q^2+q+1)$	+
23	$\omega_1\omega_4\omega_6\omega_3\omega_2\omega_5$	12	12	Z_{12}	$(q^2+q+1)(q^4-q^2+1)$	+
24	$\omega_1\omega_4\omega_{14}\omega_3\omega_2\omega_6$	9	9	Z_9	(q^6+q^3+1)	+
25	$\omega_1\omega_4\omega_{14}\omega_3\omega_2\omega_{31}$	6	72	$Z_3 \times SL_2(3)$	$(q^2-q+1) \times (q^4+q^2+1)$	+

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