Aspects of Growth in Baumslag-Solitar Groups

Groups St Andrews in Birmingham
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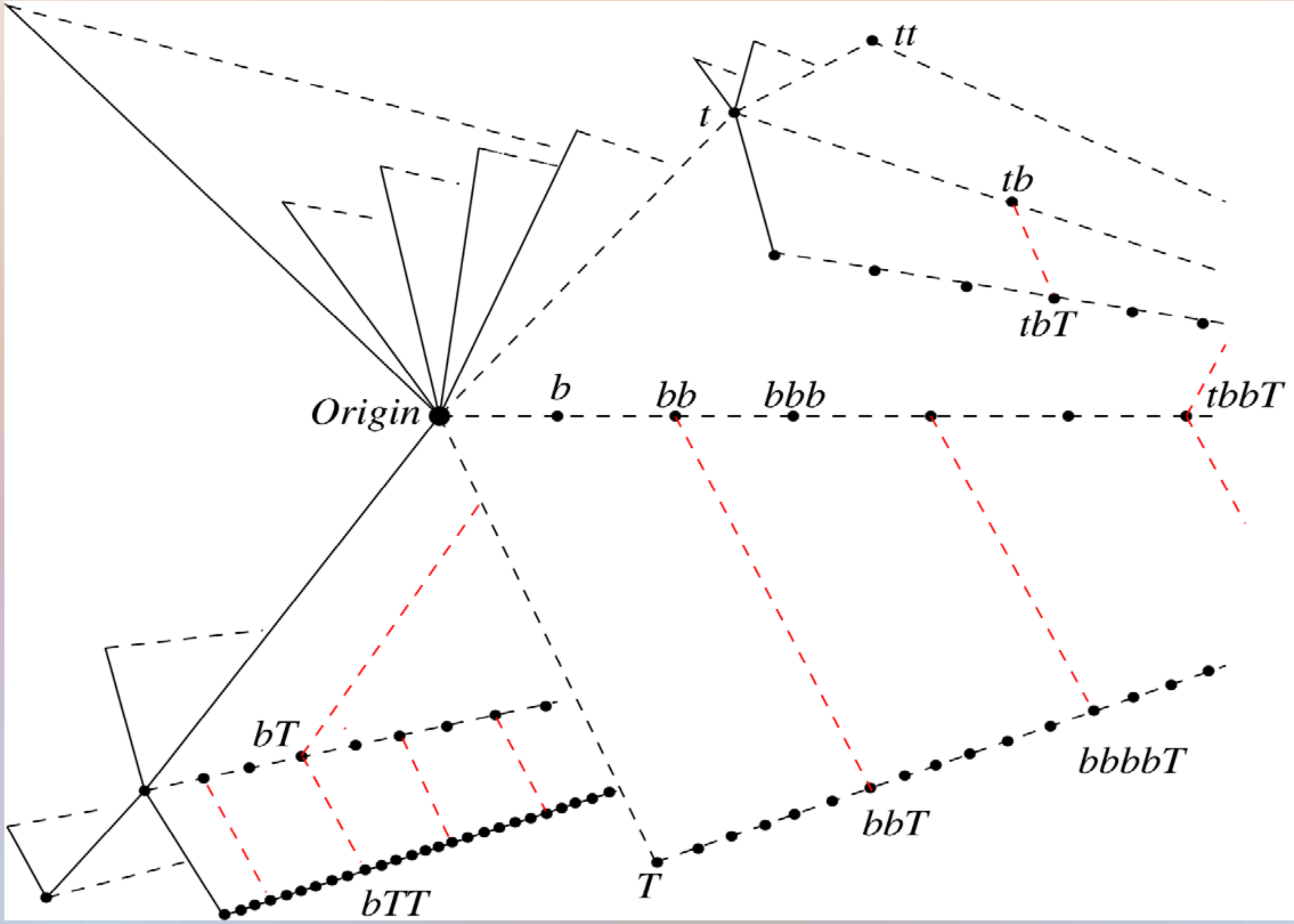
Prelude: what are the obstructions to understanding Baumslag-Solitar groups?

First issue for group theorists is the mix of positive and negative curvature.

Second issue is number theory. In BS(p, q) the relationship between p and q matters. For $p = 1$ the group is solvable. For $p = q$ the group is automatic. When p divides q, there is still a semblance of order. Chaos reigns when $p \nmid q$.

This is a (partial) Cayley 2-complex for the 2,6 Baumslag-Solitar group. Topologically it is a tree cross a line. The main line is called the *horocyclic subgroup*.



The *growth series* for a group G with respect to a specific finite generating set is a formal generating function

Summing over $n \geq 0$, let $\mathcal{S}(z) = \sum \sigma(n)\, z^n$

where $\sigma(n)$ denotes the number of group elements whose word metric length is n.

The *exponent of growth* is $\omega_{\mathcal{S}} = \lim_{n\to\infty} \sqrt[n]{\sigma(n)}$, where the limit exists by Fekete's Lemma.

$\omega_{\mathcal{S}}$ is the reciprocal of the radius of convergence for $\mathcal{S}(z)$.

If a group G is the direct product of finitely generated subgroups T, B then in terms of generating functions:

$$\mathcal{S}(z) = \sum \sigma(n)\, z^n = \mathcal{T}(z)\mathcal{B}(z)$$

the usual product of generating functions

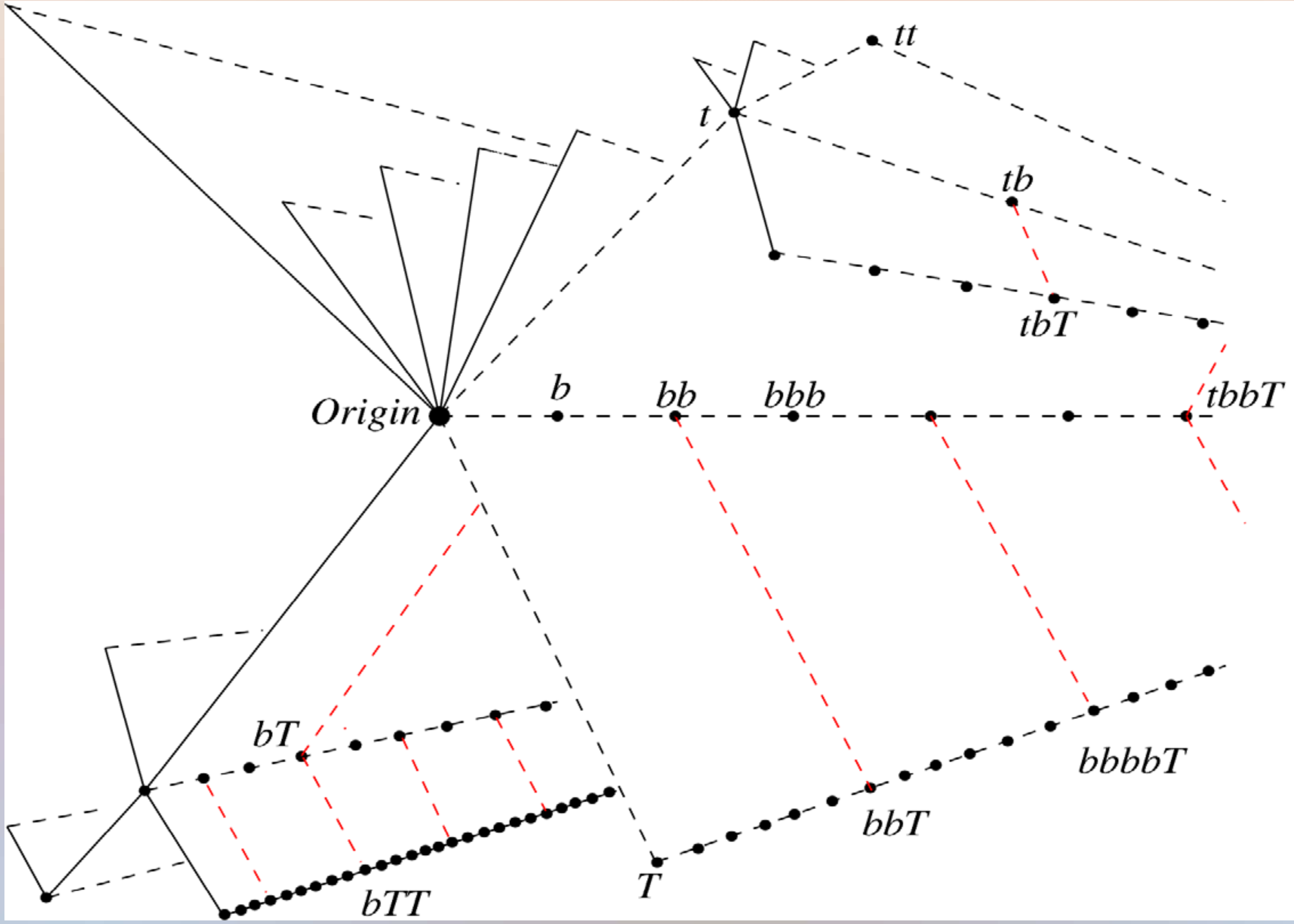
where $\sigma(n)$ can be written as the convolution sum

$$\sigma(n)= \sum \tau(k)b(n-k)$$

with the sum index going from *0* to *n*.

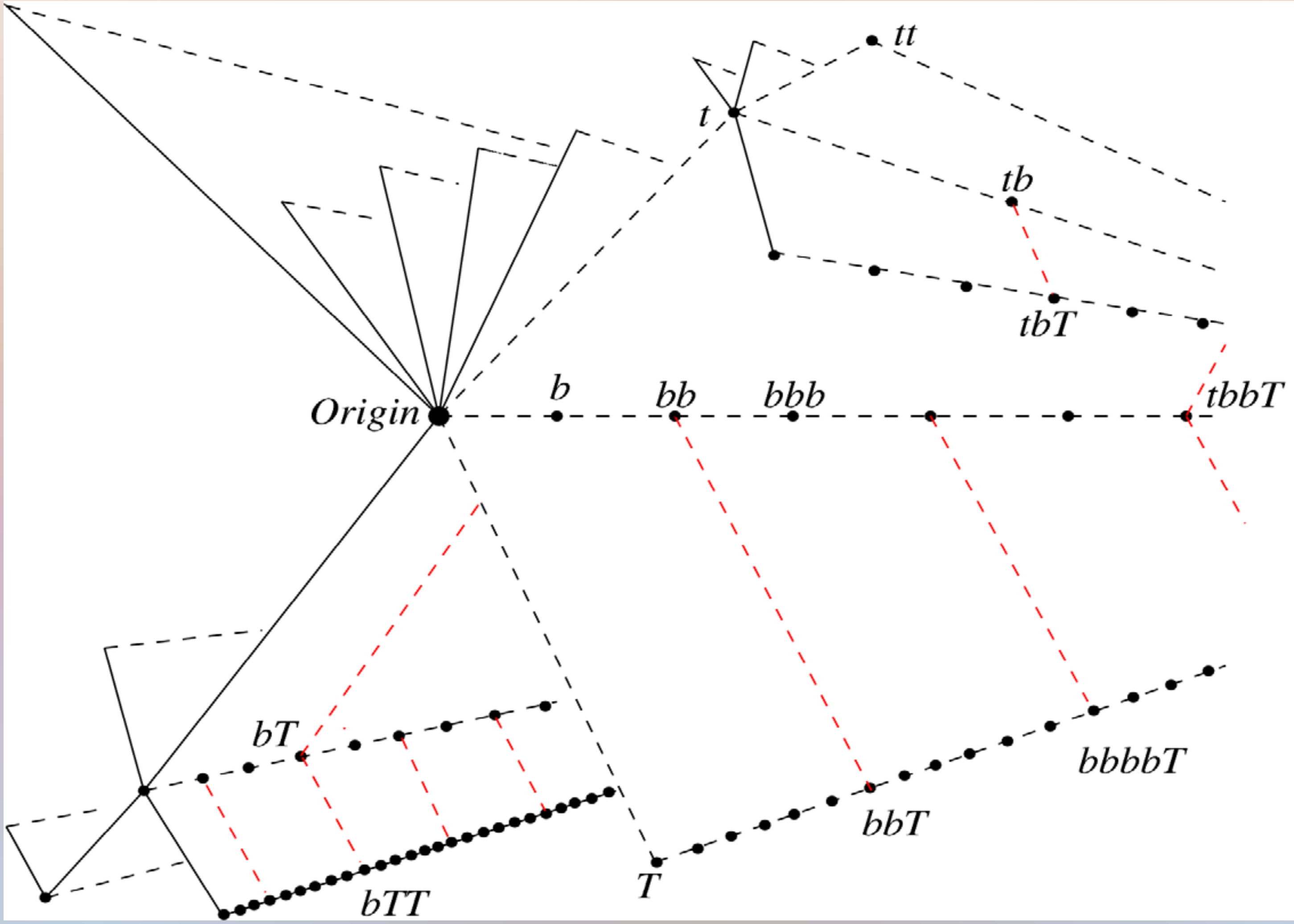
And evidently (barring miraculous cancellations) the exponent of growth $\omega_{\mathcal{S}}$ coincides with the that of the larger of the two factors.

Although topologically the 2-complex is tree × line, the group is ***not*** an algebraic product $T \times Z$ as evidenced by the distortion of the horocyclic cosets*

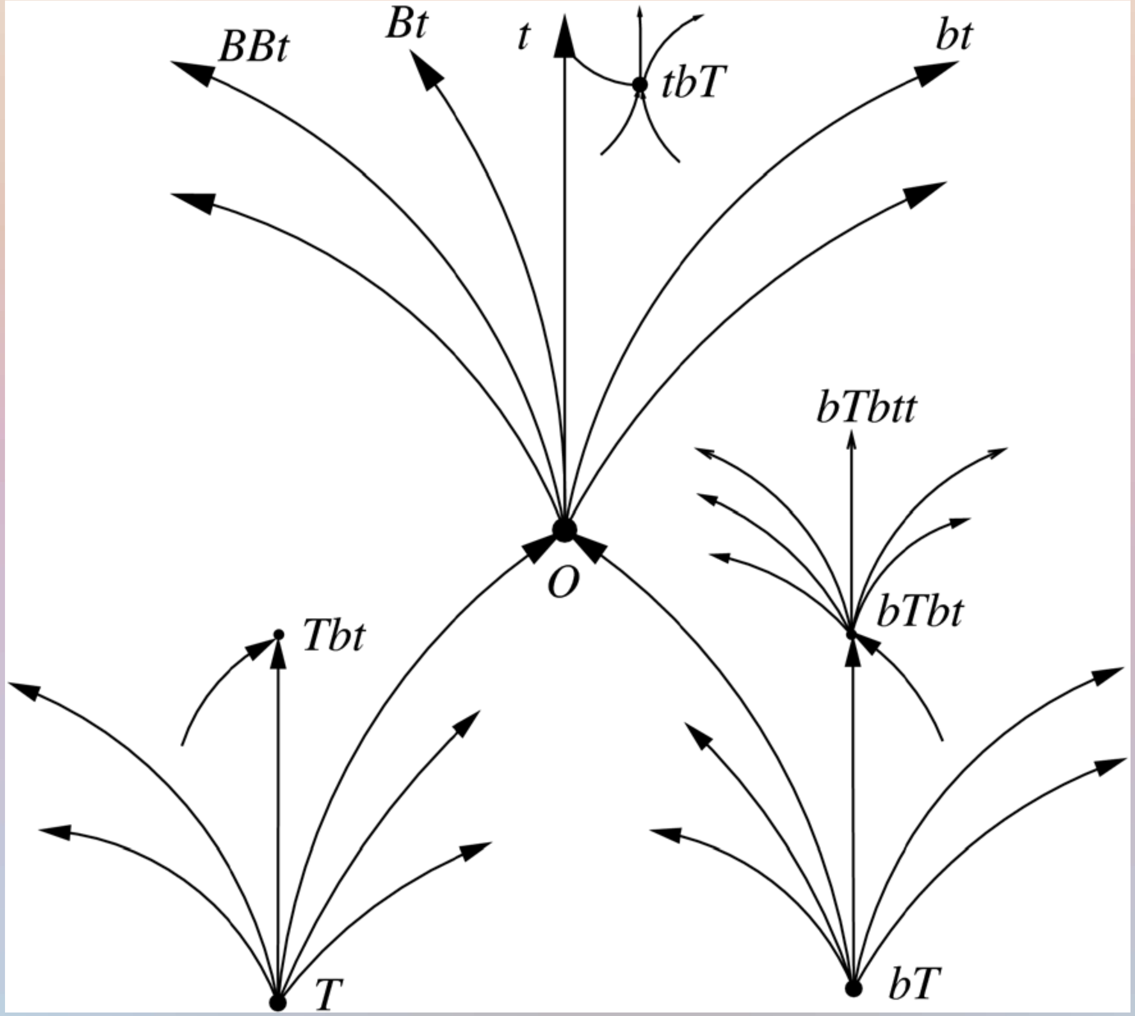


*growth of the horocyclic subgroup was the topic of my GSA 2005 presentation

So the growth function of the group is not a product of two subgroup growth functions. But it *can* be viewed as a *modified* convolution product….



Project the Cayley graph so the line components vanish. This gives a rooted tree (the Bass-Serre tree) with edge weights. If we know all the weights, we can compute the growth of the tree.



The growth series for the Bass-Serre tree is another formal generating function*

Summing over $n \geq 0$**, let** $\mathcal{T}(z) = \sum \tau(n)\, z^n$

where $\tau(n)$ **denotes the number of tree nodes whose (weighted) distance from the root is *n*. Note that sequence** $\tau(n)$ **is increasing, so by Pringsheim's Theorem, the dominant singularity for** $\mathcal{T}(z)$ **is positive and equals its radius of convergence.**

The exponent of growth for the tree is $\omega_{\mathcal{T}} = \limsup_{n>0} \sqrt[n]{\tau(n)}$

which is the reciprocal of the radius of convergence for $\mathcal{T}(z)$**.**

* the growth series for the tree was the topic of my GSA 2013 presentation

At GSA 2005, we conjectured the equality $\omega_{\mathcal{S}} = \omega_{\mathcal{T}}$ for all Baumslag-Solitar groups

Already true in the solvable and automatic cases, which have rational growth series with readily apparent tree factor in the overall generating function.

Numerical estimates suggested the conjecture holds in BS(2,3), BS(2,4), and BS(3,6).

But in the course of proving the conjecture for BS(2,4) we saw it was false in general….

Levels in the Bass-Serre tree of BS(p, q)

Our earlier picture showed distortion in horocyclic cosets.

Quasi-isometry arguments show a horizontal dilation in such cosets by a factor of $\frac{q}{p}$ for each downward tree edge.

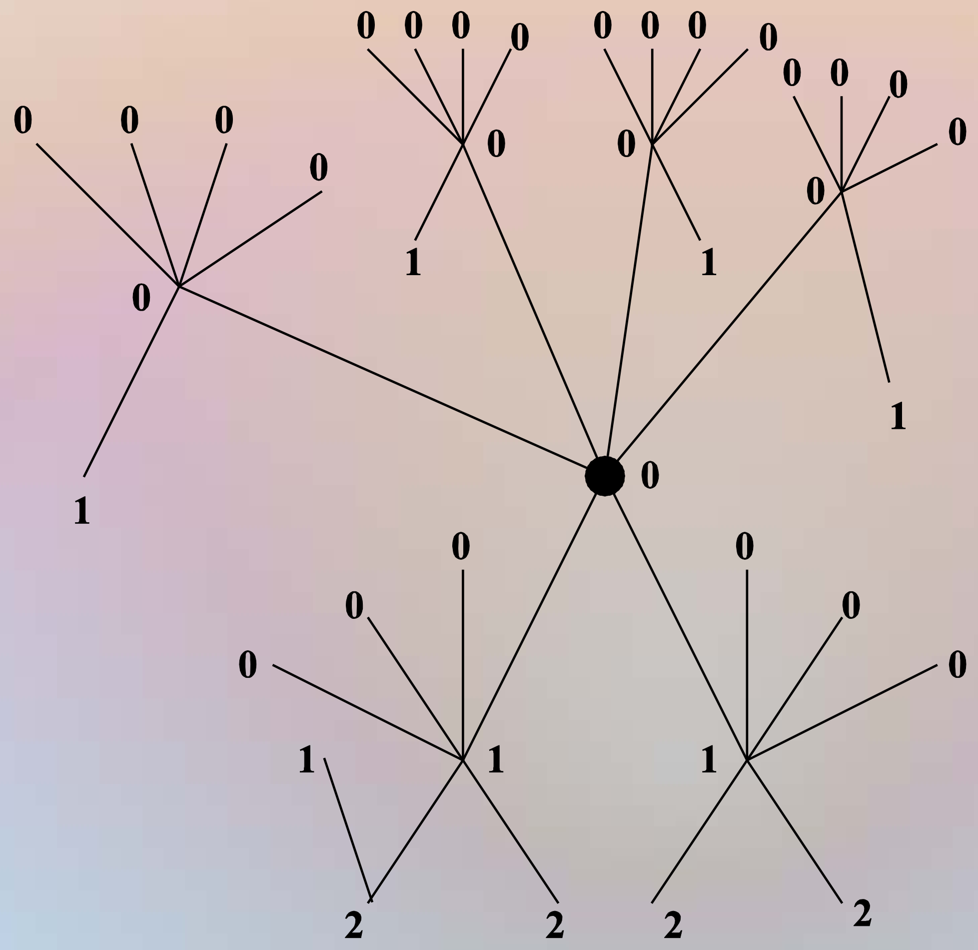
Combinatorial arguments verify this for relative growth on each particular horocyclic coset. However, combinatorial compression ceases after a certain relative height in the tree.

Partition the tree into equivalence classes based on these distortions. Label such classes as *levels*.

Levels in the Bass-Serre tree of BS(2, 4)

If level *n* > *0*, then moving up, down changes to level *n-1*, *n+1*.

If level *n* = *0*, then moving up, down changes to level *0*, *1*.



Using levels in a modified convolution for BS(2, 4)

Define the horocyclic subgroup growth series by

$$\mathscr{B}(z) = \sum b(n)\, z^n$$

Then a level ℓ coset has growth series

$$2^\ell \mathscr{B}(z) = \sum 2^\ell b(n)\, z^n$$

where in this case $2 = \frac{q}{p}$. Define $\chi(n,\ell)$ as the number of level ℓ nodes in the tree whose distance to the root is n. Evidently

$$\tau(n) = \sum \chi(n,\ell)$$

where the sum is over all levels from 0 to n.

Using levels in a modified convolution for BS(2, 4)

So instead of a direct product convolution

$$\sigma(n) = \sum \tau(k) b(n\text{-}k) \text{ , where } 0 \le k \le n$$

we break the sum into levels and magnify the horocyclic count

$$\sigma(n) = \sum \sum \chi(n,\ell)\, 2^{\ell} b(n\text{-}k) \, .$$

The left sum has index $0 \le k \le n$ while the right sum is over levels $0 \le \ell \le n$. Let's rewrite that formula as a convolution.

$$\sigma(n) = \sum \left(\frac{1}{\tau(n)} \sum \chi(n,\ell)\, 2^{\ell}\right) \tau(n) b(n\text{-}k)$$

The modified convolution for BS(2, 4)

In the modified convolution

$$\sigma(n) = \sum \left(\frac{1}{\tau(n)}\sum \chi(n,\ell)\, 2^{\ell}\right)\ \tau(n)b(n\text{-}k)$$

call the middle parenthetical term $\varphi(n)$, and note that it is a positive correction factor > 1 for almost all n.

In terms of growth series we obtain

$$\mathcal{S}(z) = \sum \sigma(n)\, z^n \ = \ (\Phi(z) \circ \mathcal{T}(z))\mathcal{B}(z)$$

where $\Phi(z) = \sum \varphi(n)\, z^n$ and "○" is the Hadamard product.

The growth rate for BS(2, 4)

From the product relationship

$$\mathcal{S}(z) = \sum \sigma(n)\, z^n = (\Phi(z) \circ \mathcal{T}(z))\mathcal{B}(z)$$

we can examine dominant singularities to see that the radius of convergence for the group satisfies

$$\frac{1}{\omega_{\mathcal{S}}} = min\left\{\frac{1}{\omega_{\Phi}} \cdot \frac{1}{\omega_{\mathcal{T}}}, \frac{1}{\omega_{\mathcal{B}}}\right\}$$

or in terms of exponents of growth

$$\omega_{\mathcal{S}} = max\left\{\omega_{\Phi}\omega_{\mathcal{T}}, \omega_{\mathcal{B}}\right\}$$

The growth rate for BS(2, 4)

We have bounds for the constituents of

$$\omega_{\mathcal{S}} = max\{\omega_{\Phi}\omega_{\mathcal{T}}, \omega_{\mathcal{B}}\}$$

namely,

$$\omega_{\mathcal{B}} = 1.30216\ldots$$

$$\omega_{\mathcal{T}} > 2.4784$$

$$\omega_{\Phi} \geq 1$$

therefore, $\omega_{\mathcal{S}} = \omega_{\Phi}\omega_{\mathcal{T}}$

The growth rate for BS(2, 4)

In the equality $\omega_{\mathcal{S}} = \omega_{\Phi}\omega_{\mathcal{T}}$

can we sharpen our bound $\omega_{\Phi} \geq 1$?

Recall our correction factor

$$\varphi(n) = \frac{1}{\tau(n)} \sum \chi(n,\ell)\, 2^{\ell}$$

where we sum over levels $0 \leq \ell \leq n$.

Evidently $\chi(n,\ell)$ cannot grow strictly faster than $\tau(n)$ but

we know* $\chi(n,\ell) = \Theta(\xi^{n-\ell-1})$ for some fixed base $2 < \xi \leq \omega_{\mathcal{T}}$

*derived via long, involved estimates using recursions

The growth rate for BS(2, 4)

Thus our correction factor becomes

$$\varphi(n) = \frac{1}{\tau(n)}\sum \Theta(\xi^{n-\ell-1})\, 2^{\ell}$$

and we can compute ω_{Φ} by ignoring sub-exponential terms:

$$\omega_{\Phi} = \limsup_{n>0} \sqrt[n]{\frac{1}{\tau(n)}\sum_{\ell=0}^{n} \xi^{n-1-\ell}\, 2^{\ell}}$$

$$= \limsup_{n>0} \sqrt[n]{\frac{\xi^{n}}{\tau(n)}} \cdot \sqrt[n]{\frac{1}{\xi}} \cdot \sqrt[n]{\sum_{\ell=0}^{n} \frac{2^{\ell}}{\xi^{\ell}}}$$

$= \dfrac{\xi}{\omega_{\mathcal{T}}} \cdot 1 \cdot 1 \le 1$, but *a priori*, $\omega_{\Phi} \ge 1$

so $\omega_{\mathcal{S}} = \omega_{\Phi}\omega_{\mathcal{T}} = 1 \cdot \omega_{\mathcal{T}}$

the growth rates of the tree and group are the same

Generalizations to BS(p, q) where p|q

Our methods extend, with a few modifications, to BS(n, $2n$). In particular the modified convolution

$$\sigma(n) = \sum \left(\frac{1}{\tau(n)}\sum \chi(n,\ell)\, 2^{\ell}\right) \tau(n)b(n-k)$$

remains valid, and our asymptotic estimates easily generalize. The group and Bass-Serre tree grow at the same rate.

On the other hand, for BS(n, kn) with $k \geq 3$, the horocyclic dilation factor of k exceeds the growth rate of BS(n, kn).

Our earlier computation shows that the n^{th} root of

$$\left(\frac{1}{\tau(n)}\sum \chi(n,\ell)\, k^{\ell}\right)$$

tends towards $\frac{k}{\omega_S} > 1$. So the group grows faster than the tree.

Generalizations to BS(p, q) where $p \nmid q$

What works when p fails to divide q? Levels and the dilation of horocycles by $\frac{q}{p}$ based on level remains valid, being a result of quasi-isometry. So a modified convolution idea appears valid.

But nothing else works! There are apparently no verifiable recursions for counting or estimating $\chi(n,\ell)$. The number-theoretic difficulties seem insurmountable.

Nevertheless, we conjecture that

$$\omega_{\mathcal{S}} = \omega_{\mathcal{T}}$$

remains valid for BS(p, q) whenever $q < 2p$.

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Thank you!