OneRelator Groups: An Overview

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joint work with Gilbert Baumslag and Gerhard Rosenberger

In memory of Gilbert Baumslag

One-relator groups have always played a fundamental role in combinatorial group theory. This is true for a variety of reasons. From the viewpoint of presentations they are the simplest groups after free groups which they tend to resemble in structure. Secondly as a class of groups they have proved to be somewhat amenable to study. However most importantly is that they arise naturally in the study of low-dimensional topology, specifically as fundamental groups of two-dimensional surfaces. At Groups St Andrews in 1985 Gilbert Baumslag gave a short course on one-relator groups which provided a look at the subject up to that point. In this talk we update the massive amount of work done over the past three decades. We look at the important connections with surface groups and elementary theory, and describe the surface group conjecture and the Gromov conjecture on surface subgroups.

We look at the solution by D. Wise of Baumslag's residual finiteness conjecture and discuss a new Baumslag conjecture on virtually free-by-cylic groups. We examine various amalgam decompositions of one-relator groups and what are called the Baumsag-Shalen conjectures. We then look at a series of open problems in one-relator group theory and their status. Finally we introduce a concept called plainarity based on the Magnus breakdown of a one-relator group which might provide a systematic approach to the solution of problems in one-relator groups.

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SURFACE GROUPS, HYPERBOLIC GROUPS AND ELEMENTARY THEORY

Much of the theory of one-relator groups, as well as much of combinatorial group theory in general, has been motivated by the properties of surface groups. This was written about in detail by Ackermann, Fine and Rosenberger (Groups St. Andrews). As new ideas such as hyperbolic groups and elementary free groups arose in group theory the important ties to surface groups continued. In this section we discuss some important results on surface groups most relevant to these new developments.

Recall that an orientable **surface group** S_g is the fundamental group of an orientable compact surface of genus g. Such a group has a one-relator presentation

$$S_g = \langle a_1, b_1, \dots, a_g, b_g; [a_1, b_1]...[a_g, b_g] = 1\rangle \text{ with } g \geq 1$$

A nonorientable surface group N_G is the fundamental group of an nonorientable compact surface of genus g. Such a group also has a one-relator presentation, now of the form

$$S_g = \langle a_1, a_2, \dots, a_g, ; a_1^2 a_2^2 \cdots a_g^2 = 1\rangle \text{ with } g \geq 1$$

Fricke and Klein proved that in the orientable case these groups have faithful representations in $PSL(2,\mathbb{C})$. It follows from a theorem of Mal'cev that each is residually finite. Recall that a group G is residually finite if given any element $g \in G$, $g \neq 1$, there exists a normal subgroup N of finite index in G such that $g \notin N$. It follows that G has a solvable word problem.

Max Dehn proved that the fundamental group of an orientable compact surface of genus $g \geq 2$ has a solvable word problem by showing that if any cyclically reduced word w is equal to 1 in S_g then more than half u of one of the cyclic conjugates uv^{-1} of $[a_1, b_1] \dots [a_g, b_g]$ or its inverse occurs in w. On replacing u by v in w, the resultant *shorter* word w' is also equal to 1 in S_g and so the process can be repeated, ultimately leading to a proof that w is equal to 1 in S_g. This algorithm is called **Dehn's algorithm**. Hyperbolic groups can be defined as those groups with a finite presentation where Dehn's algorithm solves the word problem. Dehn solved the conjugacy problem for the S_g in a similar manner; again a similar argument can be used to solve the conjugacy problem for every hyperbolic group. The class of hyperbolic groups is contained in a somewhat wider class, the class of automatic groups.

CYCLICALLY PINCHED AND CONJUGACY PINCHED ONE-RELATOR GROUPS

A **cyclically pinched one-relator group** is a group with a finite presentation of the form

$$G = F_1 \underset{U=V}{\star} F_2$$

where F_1, F_2 are free groups and U, V are nontrivial words in the respective free groups. Hence any orientable surface group of genus $g \geq 2$ falls in the larger class of cyclically pinched one-relator groups.

A **conjugacy pinched one-relator group** is the HNN analog of a cyclically pinched one-relator groups. This is a group with a finite presentation of the form

$$G =< t, F; t^{-1}Ut = V >$$

where F is a free groups and U, V are nontrivial elements in F. A surface group with $g \geq 2$ can also be expressed as a conjugacy pinched one-relator group.

CYCLICALLY PINCHED AND CONJUGACY PINCHED ONE-RELATOR GROUPS

Cyclically pinched and conjugacy pinched one-relator groups share many general properties with surface groups. This is especially true with linearity results, that is results also shared by linear groups. Wehfritz showed that a cyclically pinched one-relator group where neither U nor V are proper powers has a faithful representation over a commutative field and is hence linear. Using a result of Shalen and generalized by Fine and Rosenberger, if neither U nor V is a proper power then a cyclically pinched one relator group has a faithful representation in $PSL(2, \mathbb{C})$. Further under the same conditions Fine, Kreuzer and Rosenberger [FKR] showed that there is faithful representation in $PSL(2, \mathbb{R})$. In particular cyclically pinched one-relator groups are residually finite and coherent, that is finitely generated subgroups are finitely presented, a result originally due to Karrass and Solitar. We summarize many of these.

Theorem Let G be a cyclically pinched one-relator group. Then

(1) G is residually finite (G.Baumslag)

(2) G has a solvable conjugacy problem (S.Lipschutz) and is conjugacy separable (J.Dyer)

(3) G is subgroup separable (Brunner,Burns and Solitar)

(4) If neither U nor V is a proper power then G has a faithful representation over some commutative field (Wehrfritz).

(5) If neither U nor V is a proper power then G has a faithful representation in $PSL_2(\mathbb{C})$ (Fine,Rosenberger) and $PSL(2, \mathbb{R})$ (Fine,Kreuzer and Rosenberger)

(6) If either U or V is not a proper power and U, V are quasi-convex subgroups of their respective factors then G is hyperbolic. (Bestvinna and Feign, Juhasz and Rosenberger, Kharlampoviuch and Myasnikov)

(7) If neither U nor V is in the commutator subgroup of its respective factor then G is free-by-cyclic (Baumslag, Fine, Miller and Troeger).

(8) If G is not isomorphic to $\langle a, b; a^2 = b^2 \rangle$, then G is SQ-universal, in particular G contains a nonabelian free group (Sacerdote and Schupp).

Recall that a group G is **SQ-universal** if every countable group can be embedded as a subgroup of a quotient of G.

SQ-universality is one measure of largeness for an infinite group .

Rosenberger using Nielsen cancellation, has given a positive solution to the isomorphism problem for cyclically pinched one-relator groups, that is, he has given an algorithm to determine if an arbitrary one-relator group is isomorphic or not to a given cyclically pinched one-relator group.

Theorem (Rosenberger) The isomorphism problem for any cyclically pinched one-relator group is solvable; given a cyclically pinched one-relator group G there is an algorithm to decide in finitely many steps whether an arbitrary one-relator group is isomorphic or not to G.

Using the solvability of the isomorphism problem for all hyperbolic groups Dahmani and Guiradel proved the solvability of the isomorphism problem for all one-relator groups with torsion. This had been done earlier by S.Pride for 2-generator one-relator groups with torsion.

Theorem

(Dahmani and Guiradel) The isomorphism problem is solvable for one-relator groups with torsion.

Conjugacy pinched one-relator groups are the HNN analogs of cyclically pinched one-relator groups

Groups of this type arise in many different contexts and share many of the general properties of the cyclically pinched case. However many of the proofs become tremendously more complicated in the conjugacy pinched case than the cyclically pinched case. Further in most cases additional conditions on the associated elements U and V are necessary. To illustrate this we state a result (Fine, Röhl and Rosenberger) which gives a partial solution to the isomorphism problem for conjugacy pinched one-relator groups.

Theorem (Fine, Röhl and Rosenberger) Let $G =< a_1, ..., a_n, t; tUt^{-1} = V >$ be a conjugacy pinched one-relator group and suppose that neither U nor V is a proper power in the free group on $a_1, ..., a_n$. Suppose further that there is no Nielsen transformation from $\{a_1, ..., a_n\}$ to a system $\{b_1, ..., b_n\}$ with $U \in \{b_1, ..., b_{n-1}\}$ and that there is no Nielsen transformation from $\{a_1, ..., a_n\}$ to a system $\{c_1, ..., c_n\}$ with $V \in \{c_1, ..., c_{n-1}\}$. Then:

(1) G has rank $n + 1$ and for any minimal generating system for G there is a one-relator presentation.

(2)The isomorphism problem is solvable

SURFACE GROUPS AND ELEMENTARY THEORY

The groups S_g and N_g are also heavily involved in the elementary theory of groups.

Definition

A group G is **residually free** if for each non-trivial $g \in G$ there is a free group F_g and an epimorphism $h_g : G \to F_g$ such that $h_g(g) \neq 1$. Equivalently for each $g \in G$ there is a normal subgroup N_g such that G/N_g is free and $g \notin N_g$.
The group G is **fully residually free** provided to every finite set $S \subset G \setminus \{1\}$ of non-trivial elements of G there is a free group F_S and an epimorphism $h_S : G \to F_S$ such that $h_S(g) \neq 1$ for all $g \in S$.

A result of G. Baumslag [GB 2] showed that each S_g is residually free. Combining this with a result of B. Baumslag [BB 1] we get that further each S_g is fully residually free.

Theorem

For all $g \geq 1$ the surface group S_g of genus g is fully residually free.

Gilbert Baumslag gave a more general result. If F is a nonabelian free group and $u \in F$ is a nontrivial element which is neither primitive nor a proper power then the one-relator group K given by

$$K = F \underset{u=\overline{u}}{\star} \overline{F}$$

where $\overline{F}$ is an identical copy of F and $\overline{u}$ is the corresponding element to u in $\overline{F}$, is called a **Baumslag double**.

Theorem

(G. Baumslag) Any Baumslag double is fully residually free.

The class of finitely generated fully residually free groups were introduced in a different direction by Sela in his proof of the Tarski problems. In Sela's approach these groups appear as limits of homomorphisms of a group G into a free group. In this guise they are called **limit groups**. Therefore a limit group is a finitely generated fully residually free group. The paper by Bestvina and Feighn on limit groups and the book by Fine, Gaglione, Myasnikov, Rosenberger and Spellman **Elementary Theory of Groups** give nice descriptions of the equivalence of the two approaches.

Fully residually free groups are tied to logic and the elementary theory of groups in the following manner.
If G is a group, then the **universal theory** of G consists of the set of all universal sentences of L_0 true in G. We denote the universal theory of a group G by $Th_{\forall}(G)$. Since any universal sentence is equivalent to the negation of an existential sentence it follows that two groups have the same universal theory if and only if they have the same **existential theory**. The set of all sentences of L_0 true in G is called the **first-order theory** or the **elementary theory** of G. We denote this by $Th(G)$.
The Tarski conjectures, solved independently by Kharlampovich and Myasnikov and Sela , say essentially that all countable nonabelian free groups have the same elementary theory.

UNIVERSALLY FREE GROUPS

The following was well-known and much simpler.

Theorem

All nonabelian free groups have the same universal theory.

A **universally free group** G is a group that has the same universal theory as a nonabelian free group.

Gaglione and Spellman and independently Remeslennikov proved the following remarkable theorem.

Theorem

(Gaglione-Spellman, Remeslennikov) Let G be a finitely generated nonabelian group. Then G is fully residually free if and only if G is universally free.

Subsequently Myasnikov and Remeslennikov showed that finitely generated fully residually free groups are precisely the finitely generated subgroups of the free exponential group $F^{\mathbb{Z}[x]}$. A group is **coherent** if all finitely generated subgroups are also finitely presented. From the result of Myasnikov and Remeslennikov, Kharlampovich and Myasnikov and independently Sela proved that each fully residually free group is coherent.

ELEMENTARY FREE GROUPS

An **elementary free group** is a group having the same elementary theory as the nonabelian free groups. Clearly the class of elementary free groups contains the class of universally free groups and hence the fully residually free groups. The proofs of both Kharlampovich and Myasnikov and Sela completely describe the class of elementary free groups which extends beyond the free groups themselves. The surface groups S_g with genus $g \geq 2$ are the primary examples of nonfree elementary free groups.

Theorem

The orientable surface group S_g with $g \geq 2$ and the nonorientable surface groups T_g with $g \geq 4$ are elementary free.

This provides an interesting and powerful technique to prove nontrivial results in surface groups. These have been dubbed *something for nothing results*. In particular any first order result on nonabelian free groups is true in any elementary free groups and in particular a surface group.

Magnus proved the following often used theorem in free groups.

Theorem

(Magnus) Let F be a nonabelian free group and $R, S \in F$. Then if $N(R) = N(S)$, it follows that R is conjugate to either S or S^{-1}. Here $N(g)$ denotes the normal closure in F of the element g.

J. Howie and independently O. Bogopolski and Bogopolski and V.Sviridov gave a proof of this result for surface groups. Howie's proof was for orientable surface groups while Bogopolski and Sviridov also handled the nonorientable case. That is Magnus's theorem holds if the free group F is replaced by a surface group of appropriately high genus. Their proofs were nontrivial and Howie's proof used the topological properties of surface groups. Howie further developed, as part of his proof of Magnus' theorem for surface groups, a theory of one-relator surface groups. These are surface groups modulo a single additional relator. Bogopolski and Bogopolski-Sviridov proved in addition that Magnus's Theorem holds in even a wider class of groups.

Fine, Gaglione, Rosenberger and Spellman and Gaglione, Lipschutz and Spellman determined that Magnus's result is actually a first-order theorem on nonabelian free groups and hence from the theorems concerning the solution of the Tarski problems it holds automatically in all elementary free groups. In particular Magnus' theorem will hold in surface groups, both orientable and nonorientable of appropriate genus.

Theorem

(FGLRS) Let G be an elementary free group and $R, S \in G$. Then if $N(R) = N(S)$ it follows that R is conjugate to either S or S^{-1}.

We mention the following two corollaries which extend Magnus's Theorem to surface groups and recover the results of Howie, Bogopolski and Bogopolski-Sviridov.

Corollary

(Hpowie, Bogopolski, Bogopolski-Sviridov)) Let S_g be an orientable surface group of genus $g \geq 2$. Then S_g satisfies Magnus's theorem, that is if $u, v \in S_g$ and $N(u) = N(v)$ it follows that u is conjugate to either v or v^{-1}.

Corollary

([Bogopolsji and Sviridov) Let N_g be a nonorientable surface group of genus $g \geq 4$. Then N_g satisfies Magnus's theorem, that is if $u, v \in N_g$ and $N(u) = N(v)$ it follows that u is conjugate to either v or v^{-1}. The genus $g \geq 4$ is essential here.

To prove Magnus's theorem in elementary free groups we show that Magnus's theorem is actually a first-order result in nonabelian free groups. Since it is known to be true in nonabelian free groups it will then from the solution to the Tarski problems be true in any elementary free group.

Magnus's theorem can be given by a sequence of elementary sentences of the form.

$$\{\forall R, S \in G, \forall g \in G \exists g_1, ..., g_t, h_1, ..., h_k\}$$

$$(g^{-1}Rg = g_1^{-1}S^{\pm 1}g_1...g_t^{-1}S^{\pm 1}g_t) \wedge (g^{-1}Sg = h_1^{-1}R^{\pm 1}h_1...h_k^{-1}R^{\pm 1}h_k)$$

$$\implies \{\exists x \in G(x^{-1}Rx = S \vee x^{-1}Rx = S^{-1})\}.$$

Magnus's theorem is therefore a first-order result and the theorem follows.

It follows that any elementary free group and hence surface groups of the appropriate genus satisfy Magnus's theorem. This recovers the results of Howie and Bogopolski. Actually more is true. An examination of the sentences capturing that Magnus's theorem is first-order shows that the sentences are universal-existential. Hence the theorem holds in the almost locally free groups of Gaglione and Spellman.

Many other nontrivial results on surface groups can be proved in this manner. Further it can be proved that elementary free groups satisfy a collection of properties that are not first order. These results were done by Fine, Gaglione, Rosenberger and Spellman and in particular hold in the class of surface groups. For example, all elementary free groups are hyperbolic and stably hyperbolic. Further they are all Turner Groups, that is they have test elements and satisfy Turner's retraction Theorem

THE SURFACE GROUP CONJECTURE

Related to the problem of discerning one-relator groups by the form of their relator is the **surface group conjecture**. In the Kourovka notebook Melnikov proposed the following problem. Suppose that G is a residually finite non-free, non-cyclic one-relator group such that every subgroup of finite index is again a one-relator group. Then G is a surface group.

As asked by Melnikov the answer is no. Recall that the Baumslag-Solitar groups $BS(m,n)$ are the groups

$$BS(m,n) = < a, b; a^{-1}b^m a = b^n > .$$

If $|m| = |n|$ or either $|m| = 1$ or $|n| = 1$ these groups are residually finite. They are Hopfian if $|m| = 1$ or $|n| = 1$ or m and n have the same prime factors. In all other cases they are non-hopfian. If either $|m| = 1$ or $|n| = 1$ every subgroup of finite index is again a Baumslag-Solitar group and therefore a one-relator group. It follows that besides the surface groups the groups $BS(1,m)$ also satisfy Melnikov's question. We then have the following conjecture.

Surface Group Conjecture A Suppose that G is a residually finite non-free, non-cyclic one-relator group such that every subgroup of finite index is again a one-relator group. Then G is either a surface group or a Baumslag-Solitar group $B(1,m)$ for some integer m.

We note that the groups $B(1,1)$ and $B(1,-1)$ are surface groups. In surface groups, subgroups of infinite index must be free groups and there are noncyclic free groups. This is not true in the groups $BS(1,m)$. To avoid the Baumslag-Solitar groups, Surface Group Conjecture A, was modified to:

Surface Group Conjecture B Suppose that G is a non-free, non-cyclic one-relator group such that every subgroup of finite index is again a one-relator group and every subgroup of infinite index is a free group and G contains nonabelian free groups as subgroups of infinite index. Then G is a surface group.

Using the structure theorem for fully residually free groups in terms of its JSJ decomposition Fine, Kharlampovich, Myasnikov, Remeslennikov and Rosenberger made some progress was made on these conjectures. Finally Ciobanu, Fine and Rosenberger building on work of H. Wilton settled the surface group conjecture if G is assumed to be either a cyclically pinched one-relator group of a conjugacy pinched one-relator group.

We say that a group G satisfies **Property IF** if every subgroup of infinite index is free. Recall that the one-relator presentation for a surface group allows for a decomposition as a cyclically pinched one-relator group or conjugacy pinched one-relator group. In particular the following results were proved.

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Theorem

Suppose that G is a finitely generated fully residually free group with property IF. Then G is either a free group or a cyclically pinched one relator group or a conjugacy pinched one relator group.

Corollary

Suppose that G is a finitely generated fully residually free group with property IF. Then G is either free or every subgroup of finite index is freely indecomposable and hence a one-relator group.

The following modified version of the surface group conjecture was given.

Surface Group Conjecture C Suppose that G is a finitely generated non-free freely indecomposable fully residually free group with property IF. Then G is a surface group.

Using the following result of H.Wilton combined with results of Guildenhuys, Kharlampovich and Myasnikov and the Karrass-Solitar subgroup theorems for amalgamations Ciobanu, Fine and Rosenberger settled Surface group Conjecture C and the general conjecture for cyclically pinched and conjugacy pinched one-relator groups.

Theorem

(Wilton) Let G be a hyperbolic one-ended cyclically pinched one-relator group or a hyperbolic one-ended conjugacy pinched one-relator group. Then either G is a surface group, or G has a finitely generated non-free subgroup of infinite index.

Ciobanu, Fine and Rosneberger proved that surface group conjecture C is true.

Theorem

Suppose that G is a finitely generated nonfree freely indecomposable fully residually free group with property IF. Then G is a surface group. That is Surface Group Conjecture C is true.

Thus fully residually free and Property If completely characterize surface groups.

Theorem

G is a surface group if and only if G is finitely generated, nonfree, indecomposable, fully residually free and satisfies Property IF.

The main result of Ciobanu, Fine and Rosenberger is that the Surface Group Conjecture is true if G is a cyclically pinched or conjugacy pinched one-relator.

Theorem

(1) Let G be a cyclically pinched one-relator group satisfying property IF. Then G is a free group or a surface group.

(2) Let G be a conjugacy pinched one-relator group satisfying property IF. Then G is a free group, a surface group or a solvable Baumslag-Solitar group.

GROMOV'S SURFACE GROUP CONJECTURE

A conjecture of Gromov states that a one-ended word hyperbolic group must contain a subgroup isomorphic to the fundamental group of a closed hyperbolic surface. Kim and Oum proved that this is true for any one-ended Baumslag double if (1) the free group has rank 2 or (2) every generator is used the same number of times in a minimal automorphic image of the amalgamating words. This builds on work of Gordon and Wilton and Kim and Wilton who gave sufficient conditions for hyperbolic surface groups to be embedded in a Baumslag double G. Fine and Rosenberger using Nielsen cancellation methods proved that a hyperbolic orientable surface group of genus 2 is embedded in a hyperbolic Baumslag double if and only if the amalgamated word W is a commutator, that is $W = [U, V]$ for some elements $U, V \in F$. Further G contains a nonorientable surface group of genus 4 if and only if $W = X^2Y^2$ for some $X, Y \in F$. G can contain no nonorientable surface group of smaller genus.

The main result is that of Kim and Oum.

Theorem

A one-ended Baumslag double has a hyperbolic surface subgroup if

(1) the free group has rank 2

or

(2) every generator is used the same number of times in a minimal automorphic image of the amalgamating words.

If a group contains a hyperbolic surface subgroup of genus 2 it will contain a hyperbolic surface subgroup of any genus $g \geq 2$. Using Nielsen cancellation methods Fine and Rosenberger prove the following.

Theorem

(Fine and Rosenberger) Let $G = F \underset{\{W=\overline{W}\}}{\star} \overline{F}$ *be a hyperbolic Baumslag double. Then* G *contains a hyperbolic orientable surface group of genus 2 if and only* W *is a commutator, that is* $W = [U, V]$ *for some elements* $U, V \in F$*. Further a Baumslag double* G *contains a nonorientable surface group of genus 4 if and only if* $W = X^2Y^2$ *for some* $X, Y \in F$*.*

Corollary

Let $G = F \underset{\{W=\overline{W}\}}{\star} \overline{F}$ *be a hyperbolic Baumslag double. Then* G *contains orientable surface groups of all finite genus if and only* W *is a commutator.*

THE RESIDUAL FINITENESS CONJECTURE

In 1967 Gilbert Baumslag conjectured that all one-relator groups with torsion are residually finite. There were partial results on the conjecture by Allenby and Tang using particular forms of the relator however the full conjecture was settled affirmatively by D. Wise in 2009 with a beautiful geometrically inspired proof using what he called **cube complex theory**. The details of his proof be found in his monograph **The Structure of Groups with a Quasiconvex Hierarchy**. There is also a nice summary of his methods in that can be found on the internet in a series of lectures on Baumslag's work.

Theorem

(D.Wise) If G is a one-relator group with torsion, then G is residually finite.

A group is **coherent** is every finitely generated subgroup is finitely presented. Baumslag also conjectured that one-relator groups with torsion are coherent. This was also answered affirmatively in 2005 by by D. Wise.

Theorem

(D.Wise) A one-relator group with torsion is coherent.

THE VIRTUALLY FREE-BY-CYCLIC CONJECTURE

Wise's methods use a hierarchy based on the Magnus breakdown that was also used in a paper in 2007 by Baumslag, Miller and Troeger who gave an example of a one-relator group (necessarily torsion-free) that is not residually finite showing further the complexity of one-relator group theory. In particular they proved

Theorem

([BMT]) Let $G = \langle a_1, ...a_n, ...; r = 1\rangle$ with $n \geq 2$. Suppose that w is any element in the ambient free group on $a_1, ..., a_n$ which does not commute with r. Then the group

$$G(r, w) = \langle a_a, ..., a_n; r^{r^w} = r^2\rangle$$

is a one-relator group with the same finite images as G. Further $r \neq 1$ in $G(r, w)$ and r is contained in every subgroup of finite index in $G(r, w)$. Therefore $G(r, w)$ is not residually finite.

The Virtually Free By Cyclic Conjectures

Related to the residual finiteness conjecture, Baumslag, Fine, Miller and Troeger, in a series of papers worked on the following wide ranging strong conjecture which implies the residual finitness property.

Conjecture Every one–relator group with torsion is virtually free–by–cyclic i.e., contains a subgroup of finite index which is an infinite cyclic extension of a free group.

Further the conjecture may be true for a wider class of torsion-free one-relator groups. Each surface group is free-by-cyclic and we believe that each Baumslag double is virtually free by cyclic. Since a finitely generated virtually free–by–cyclic group is residually finite, the residual finiteness theorem of Wise is a consequence of the above. Moreover finitely generated free–by–cyclic groups are also coherent (Feighn and Handel), that is, their finitely generated subgroups are finitely presented.

IBaumslag, Fine, Miller and Troeger studied the virtually free-by-cyclic structure of cyclically pinched one-relator groups, They show that it is rather well behaved. We let $\mathcal{VFC}$ denote the class of virtually free-by-cyclic groups. Then:

Theorem

The class $\mathcal{VFC}$ is closed under subgroups and free products.

Free-by-cyclic groups and virtually free-by-cyclic groups arise in many different contexts. In particular the fundamental groups of all orientable surfaces of genus $g \geq 2$ and nonorientable of genus $g \geq 3$ are free-by-cyclic. It follows that all finitely generated Fuchsian groups are virtually free-by-cyclic. In the same spirit a result of J.Howie gives sufficient conditions for the fundamental group of a 2-complex to be free-by-cyclic in terms of a Morse function.

Several results were proved by Baumslag, Fine, Miller and Troeger concerning the free-by-cyclic structure of both cyclically pinched one-relator groups and conjugacy pinched one-relator groups.

Theorem

Suppose that $G = A \underset{U=V}{\star} B$ is a cyclically pinched one-relator group. If $U \notin [A, A]$ and $V \notin [B.B]$ then G is free-by-cyclic.

Theorem

Suppose that $G =< F, t; t^{-1}Ut = V$ with $U, V \notin [F, F]$ is a conjugacy pinched one-relator group. If either $U[F, F] = V[F, F]$ or U, V are linearly independent modulo $[F, F]$ then G is free by cyclic.

These two results deal with elements not in the derived group. We can make progress when they are in the derived group in the case of **Baumslag doubles**.

Theorem

Baumslag doubles are virtually free-by-abelian. That is $G = F \underset{U=\overline{U}}{\star} \overline{F}$ *and* $H =< F, t; t^{-1}Ut = U >$ *are virtually free-by-abelian.*

There are also technical sufficient conditions so that doubles are actually virtually free-by-cyclic.

AMALGAM DECOMPOSITIONS OF ONE-RELATOR GROUPS

Baumslag and Shalen proved the following.

Theorem

Let G be a one-relator group with at least 3 generators. Then G has a nontrivial free product with amalgamation decomposition $G = A \star_H B$ where A and B are finitely generated and H is finitely generated.

An alternative proof in the more general case of one-relator products of cyclics with torsion was given by Fine, Levin and Rosenberger using the dimension of the character variety Recently Benyash-Krivets extended this to show that all noncyclic one-relator groups with torsion are nontrivial free products with amalgamation.

Unfortunately in these decomposition results very little is known about the exact nature of the factors and this would have to be studied to gain more information about one-relator groups. Fine and Peluso call a free product with amalgamation decomposition of a group G with finitely generated factors a **Baumslag-Shalen Decomposition**.

BS Conjecture 1 Let G be a one-relator group. Then G has a BS-decomposition

$$G = A \underset{H}{\star} B$$

where the factors A and B are either one-relator groups or free groups and H is free,

BS Conjecture 2 (The Strong BS Conjecture) Let G be a one-relator group. Then if

$$G = A \underset{H}{\star} B$$

is a BS-decomposition of G with H free then the factors A and B are either one-relator groups or free groups.
The strong BS Conjecture was proved up to homology by Fine and Peluso..

The existence of these amalgam decompositions for one-relator groups can be used to prove that many of them are SQ-universal. From a result of Sacerdote and Schupp a torsion-free one-relator group with at least three generators is SQ-universal. Recall again that a group G is **SQ-universal** if every countable group can be embedded as a subgroup of a quotient of G. SQ-universality is one measure of largeness for an infinite group . Q-universal. From the Baumslag-Shalen decomposition theorem above together with a result of Lossov it follows that a one-relator group with torsion and at least three generators is SQ-universal. A recent result of Benyash-Krivets showed that all noncyclic one-relator groups with torsion are nontrivial free products with amalgamation. Combining this with the result of Lossov gives that any noncyclic one-relator group with torsion is SQ-universal. This last result also can be deduced easily from the SQ-universality arguments for one-relator quotients of free products of cyclic groups given in Fine and Rosenberger.

Theorem

Let G be a torsion-free one-relator group with at least three

Theorem

Let G be a noncyclic one-relator group with torsion. Then G is SQ-universal.

OPEN PROBLEMS ON ONE-RELATOR GROUPS

(1)**The Standard Decision Problems**

Combinatorial group theory has always been concerned with the three major decision problems of Max Dehn; the word problem, the conjugacy problem and the isomorphism problem.

Using the Freiheitssatz and the Magnus breakdown, Magnus was able to prove that the word problem and generalized word problem are solvable for all one-relator groups. However the conjugacy problem has proved to be quite difficult. S.Lipschutz proved that the conjugacy problem is solvable for cyclically pinched one-relator groups. One-relator groups with torsion are hyperbolic. Hence they have solvable conjugacy problem. This was proved first by B.B. Newman by means of his so-called spelling theorem. Here we note that every one-relator group with torsion has a torsion-free, fully invariant subgroup of finite index (see Karrass, Magnus and Solitar). A considerable effort over many years was expended by A. Juhasz who attempted to use ideas from small cancellation theory, but with only partial success. Other partial results have been

Theorem

The conjugacy problem is solvable for both cyclically pinched one-relator groups and for one-relator groups with torsion.

The isomorphism problem is of course the most difficult of the decision problems.

Theorem

(Rosenberger) The isomorphism problem is solvable for cyclically pinched one-relator groups.

Dahmani and Guiradel proved that the isomorphism problem for hyperbolic groups with torsion is solvable. Sela had earlier proved the solvability of the isomorphism problem for torsion free hyperbolic groups. One-relator groups with torsion are hyperbolic and therefore it follows that all one-relator groups with torsion have a solvable isomorphism problem.

Theorem

(Dahmani and Guiradel) The isomorphism problem is solvable for

G.Baumslag introduced the following class of groups;

$$G_{m.n} =< a, b, t; a^{-1} = [b^m, a][b^n, t] > \text{ with } m, n \geq 1.$$

He then showed that these groups are all parafree. However Magnus and Chandler in their History of Combinatorial Group Theory mention these groups as an example of the difficulty of the isomorphism problem for one-relator groups. Up until 1993 there was no proof showing that any of the groups $G_{m,n}$ are nonisomorphic. S. Liriano using representations of $G_{m,n}$ into $PSL(2, p^k)$ showed that $G_{1,1}$ and $G_{30,30}$ are nonisomorphic. Subsequently in 1997 Fine, Rosenberger and Stille using Nielsen cancellation methods showed that the isomorphism problem is solvable for the subclass $G_{n,1}$. Further it can be decided algorithmically whether or not an arbitrary one relator group is isomorphic to $G_{n,1}$. In particular

Theorem

(Fine, Rosenberger and Stille) Let n be a natural number and $G_{n,1}$ be the groups defined above. Then
(1) the isomorphism problem for $G_{n,1}$ is solvable, that is, it can be decided algorithmically in finitely many steps whether or not an arbitrary one relator group is isomorphic to $G_{n,1}$.
(2) $G_{n,1}$ is not isomorphic to $G_{1,1}$ for $n \geq 2$
(3) If p, q are primes then

$$G_{p,1} \cong G_{q,1} \text{ if and only if } p = q$$

Further for all n the group $G_{n,1}$ is Hopfian and then every automorphism of $G_{n,1}$ is induced by an automorphism of the free group F on a, b, t. In addition the automorphism group of $G_{n,1}$ is finitely generated.

Linearity of One-relator Groups

In general it is not known which one-relator groups are linear. From the results of Poincare and formalized by Fricke and Klein it follows that surface groups are linear. This has been generalized in several ways. Wehrfritz showed that a cyclically pinched one-relator group where neither U nor V are proper powers has a faithful representation over a commutative field. This was improved upon by Shalen and generalized by Fine and Rosenberger to show that cyclically pinched one-relator groups with neither U nor V proper powers and more generally all **groups of F-type** have faithful representations in $PSL_2(\mathbb{C})$. Further under the same conditions Fine, Kreuzer and Rosenberger [FKR] showed that there is faithful representation in $PSL(2, \mathbb{R})$.

Theorem

Let G be a cyclically pinched one-relator group with neither U nor V a proper power. Then

(1) G has faithful representations in $PSL_2(\mathbb{C})$ and $PSL(2,\mathbb{R})$

(2) G is residually finite

(3) G is Hopfian

Theorem

Let G be a one-relator group with torsion. Then

(1) G has an essential representation in $PSL_2(\mathbb{C})$

(2) G is virtually torsion-free

Many of these linearity results can be extended to a wider class of groups called **groups of F-type**,as above, which generalize both cyclically pinched one-relator groups and Fuchsian groups. These were introduced by Fine and Rosenberger in and a complete description and discussion is in the book **Algebraic Generalizations of Discrete Groups**

The proof that a group is linear is a daunting task in general. The most general approach is Lubotzky's remarkable characterisation of finitely generated linear groups.

Hyperbolicity of One-relator Groups

The **Gersten conjecture** is that a torsion-free one-relator group is hyperbolic if and only if it does not contain any Baumslag-Solitar group as a subgroup. It is known that one-relator groups with torsion are hyperbolic.

Gersten has also asked whether a finitely generated one-relator group whose abelian subgroups are cyclic, is hyperbolic.

It is known that certain cyclically pinched one-relator groups and conjugacy pinched one-relator groups are hyperbolic. More generally, hyperbolicity is preserved under special amalgam constructions. Bestvina and Feighn have shown that an amalgam of two hyperbolic groups over a quaisconvex cyclic subgroup that is malnormal in at least one of the factors is still hyperbolic. This implies that certain cyclically pinched one-relator groups are hyperbolic. Kharlamapovich and Myasnikov have a more general result that the amalgam of two hyperbolic groups is again hyperbolic whenever one of the amalgamated subgroups is quasiconvex and malnormal in its respective factor. Related results were proved by Juhasz and Rosenberger for cyclically pinched one-relator groups and groups of F-type.

PLAINARITY A POTENTIAL GENERAL APPROACH

In the course of proving the Freiheitssatz, Magnus developed a technique for handling one-relator groups in general. He proved that a one-relator group can be broken down into amalgamated products of simpler one-relator groups, thereby providing an inductive mechanism for handling them. We call this whole procedure the **Magnus Breakdown**. An in depth focus on Magnus' approach will be a primary tool, though not the only one, in any exploration of a one-relator group. Usually the first step in an induction of the kind alluded to above with what we will now term **plain** one-relator groups, in accordance with the following definition.

Definition

Let $G =< a, \dots ; r >$ be a one-relator group. G is said to be **plain** if r is a power of a primitive element in the underlying free group on the given generators $a, \dots$ of G.

Moldavanski [Mo] formalized the Magnus breakdown. He used that if

$$G =< a, \ldots ; r >$$

is a one-relator group a then G can be embedded, in a very simple way, in a closely related one-relator group G_0, which often coincides with G. G_0 is an HNN-extension of a second one-relator group H; the length of the given defining relator of H is smaller than the length of r. This method was used in the Dani Wise solution of the residual finitenss conjecture.
This procedure gives rise to a series of one relator groups G_0, G_1, $\ldots$ ending up in a plain one-relator group G_d. The length d of such a series depends on a number of choices that are available at almost every step in the production of the series. We say that that G is of **plainarity** d if it has such a series of length d.

Notice that the one-relator groups of plainarity 0 are the plain ones. The plain one-relator groups are either free or the free product of a finite cyclic group and a free group. Thus they can be considered as well known. The answers to all of the problems and questions that we consider are known for such plain one-relator groups. The one-relator groups of plainarity 1 are more difficult to investigate than the ones of plainarity 0. To handle a particular problem on one-relator groups is to combine an induction on plainarity with other techniques such as Reidemeister-Schreier. Baumslag, Miller and Troeger showed how this could be done with a particular Fuchsian group in a method they call unravelling th eone-relator group.